Advances in Structural and Multidisciplinary Optimization
Proceedings of the 13th World Congress of Structural and Multidisciplinary Optimization (WCSMO13)

结构与多学科优化研究进展——第13届世界结构与多学科优化大会论文集
INTRODUCTION

The volume contains papers submitted to the WSCMO-13 conference held in Beijing in May 2019. It collects research works associated with various aspects of the optimal design of structures and multidisciplinary optimal design. The topics concern the analysis of solids, fluids or problems arising in other fields. The optimisation methods for applications and the corresponding software development in related branches of technology are also involved in the volume.

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UNCERTAINTY QUANTIFICATION AND STATISTICAL MODEL VALIDATION FOR OFFSHORE JACKET STRUCTURE PANEL GIVEN LIMITED TEST DATA AND SIMULATION MODEL

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Abstract

Due to inherent variability (i.e., aleatory uncertainty) in material properties, loading conditions, manufacturing processes, etc., simulation output responses should follow certain distributions, which are the outcome of input distribution models and simulation models. Thus, Uncertainty Quantification (UQ) using simulation-based methods is accurate only if we have (1) accurate input distribution models and (2) an accurate simulation model. However, in practical engineering applications, only limited numbers of input test data are available for modeling input distributions. Moreover, the simulation model could be biased due to assumptions and idealizations in the modeling process. Thus, the simulation model needs to be validated to correctly predict the output response of the physical system. The statistical validation of the simulation model would require large numbers of physical test data, which is extremely expensive. This study presents a computational method to obtain a target output distribution, which is a good approximation of the true output distribution given limited test data and a biased model. Using Bayesian analysis, possible candidates of output distribution are obtained, and a target output distribution is selected at the posterior median. The target output distribution is used to measure the bias of the simulation models and the surrogate models for UQ and statistical model validation. Furthermore, the cumulative distribution function of the simulation model prediction error is proposed to provide (1) the most probable model prediction error and (2) model confidence at a user-specified error level. To demonstrate the proposed statistical model validation method, a simulation model of an offshore jacket structure panel is generated using ANSYS. The output response of interest is a reaction force. The simulation model is validated using three sets of material test data and experimental test data to determine model confidence on output prediction and the validity of the simulation model. The result of statistical model validation shows how much bias the simulation model has and how much confidence engineers can have in the model prediction at a user-specified error level.

Key words: Uncertainty Quantification, Offshore Jacket Structure Panel, Statistical Model Validation, Epistemic Uncertainty, Limited Number of Test Data

1. Introduction

In manufacturing industries and the Department of Defense, Computer-Aided Engineering (CAE) simulation models are widely used to predict the outputs of a system to save product development cost and time. However, the accuracy of the simulation models and how much confidence engineers have in their simulation model prediction are persisting in problems. Furthermore, validating the performance of the model and building confidence in the model prediction requires a large number of physical test data, which is very expensive. Thus, it is desirable to be able to check the accuracy of the simulation model prediction when limited experimental data is available. It is worth noting that real-world phenomena are not deterministic, so pointwise comparison between model prediction and test data in the context of model validation is not suitable. Therefore, statistical methods that compare model prediction and physical test data are necessary due to the inherent variability. However, the challenge is that only limited numbers of input and output test data are provided in engineering applications for uncertainty quantification and model validation.

To consider those uncertainties, this paper presents a novel probabilistic method and demonstrates using the topologically optimized design of an offshore structure panel. The paper consists of the following sections. Section 2
explains how the simulation is modeled and carried out, and how experimental data is obtained. Section 3 explains the methodology for quantifying the uncertainty in the input distribution model, quantifies the uncertainty due to insufficient experimental data via the Bayesian approach, and selects target output distribution for statistical model validation. Section 4 describes the result of the statistical model validation, and section 5 summarizes the findings of the study.

2. Problem Description

2.1 Simulation Model

The optimal design of an offshore structure panel obtained by topology optimization has been used for validation [1]. Figure 1 depicts a snapshot of the Finite Element Analysis (FEA) models of the offshore structure panel using ANSYS. A constant displacement (0, 15 mm) in the vertical direction is applied at the bottom of the model while the remote displacement boundary condition is applied at the top of the model to mimic the sagging shape observed from the experiment. In the remote displacement boundary condition at the top surface, six degrees of freedom (translation and rotation in the nodal x, y, and z directions) are fixed as zero with deformable behavior. The output response of interest is the reaction force obtained from the displacement boundary condition applied at the top surface of the model. There are eight random input variables—six widths of offshore structure panel, yield strength, and Young’s modulus—and among those variables, only three input data are provided for the yield strength and Young’s modulus. The FEA is carried out under a linear static condition using a higher—order 20—node solid and 15 node wedge (prism) elements. To prevent any singularities in FEA, a total of 101K elements are utilized, and the size of elements is restricted to 0.5 mm by the ANSYS meshing tool.

![Finite Element Analysis Models](image)

Figure 1. Six widths of the panel (left), a snapshot of finite element analysis using ANSYS (middle), and a specimen in experimental testing (right)

2.2 Material Testing

To identify the variability of material properties, three samples are tested to obtain the yield strength and Young’s modulus as listed in Table 1. Each specimen is manufactured by metal 3-D printing based on Direct Metal Tooling (DMT®) at InssTek. DMT® is classified as Directed Energy Deposition (DED) technology by the American Society for Testing and Materials (ASTM) standards. In general, the DED method produces fully dense material that has excellent mechanical and fatigue properties.

<table>
<thead>
<tr>
<th>Table 1. Material Test Data</th>
</tr>
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<tbody>
<tr>
<td>Test 1</td>
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<tr>
<td></td>
</tr>
<tr>
<td>Test 2</td>
</tr>
<tr>
<td>Test 3</td>
</tr>
</tbody>
</table>

2.3 Experimental Data

To verify the performance of the optimized design of the offshore structure panel, a compression test has been performed with 1,000 kN capacity Universal Testing Machine (UTM) and a non-contact extensometer as shown on the right side of Figure 1. During the experiment, the hydraulic table located on the bottom of the UTM moves upward to apply force. A displacement loading with a constant velocity of 0.02 mm/sec is applied on the specimen. The UTM allows the deformation of the specimen at both the top and bottom since neither side is entirely fixed in
every direction. From three repeated tests, three reaction force data (5.806 kN, 6.179 kN, and 6.625 kN) have been obtained at 0.15 mm of displacement.

3. Methodology for Uncertainty Quantification and Statistical Model Validation

3.1 Input Distribution Identification

The input distribution model, which is defined by the input distribution type and the input distribution parameter, represents the variability of input random variables. To consider the uncertainty in an input distribution model induced by a limited number of input data, the Bayesian approach could be applied to produce all possible candidates of input distribution based on the limited input data. However, considering all candidates of input distribution may lead to too wide an output distribution [2]. Thus, we approximate the best input distribution type using Maximum Likelihood Estimation (MLE) given a limited number of input data. Six candidates of distribution type (Normal, Lognormal, Weibull, Gumbel, Extreme I, and Extreme II) are investigated to evaluate the likelihood value using probability density function (PDF). Distribution parameters for each distribution type are determined based on the sample mean and sample standard deviation. Among the six candidates, the one that has the maximum log-likelihood is selected as the input distribution type that best represents the given data. The mathematical formula for the log-likelihood function is given below.

$$\log L(x | \theta) = \sum_{i=1}^{n} \log f(x_i ; \theta)$$

Where $f_i$ is the input PDF, $n$ is the number of input data, $x_i$ is the input data vector, and $\theta$ is the distribution parameter vector that was transformed based on the sample mean and standard deviation. For the normal distribution, the distribution parameter vector is the same as the sample mean and standard deviation.

3.2 Uncertainty Quantification using Monte Carlo Simulation on the Dynamic Kriging Model

After approximating the input distribution model in section 3.1, the output variability (simulation output distribution) resulting from the input variability can be obtained through simulation solvers. In this study, Monte Carlo Simulation (MCS) is employed for uncertainty quantification. However, the direct use of MCS to obtain the output distribution requires a very large number of simulation runs, which is computationally expensive. To handle this issue, surrogate models are often used to reduce computational time by using a small number of simulation runs. Then, the prediction error of the surrogate model could arise. To reduce the uncertainty of the surrogate model, a very accurate dynamic Kriging (DKG) method [3-6] is utilized to evaluate simulation output responses. The DKG model has two unique features compared to other Kriging surrogate modeling methods. First, the DKG model automatically chooses the best combination of mean structure (3 candidates of polynomial order), covariance (7 candidates of correlation function) and 2 likelihood functions, to have 42 different options for surrogate models. Secondly, the DKG model uses a variance window, instead of a global domain, to reduce the number of required design of experiment (DOE) points for accurate results. The size of the variance window is determined to cover the input distribution model. In the variance window, uniformly distributed sample points are initially selected. After the initial run, DOE points are added sequentially to reduce the prediction variance of the DKG model between DOE samples until the target accuracy is achieved.

3.3 Bayesian Inference to Consider Uncertainty Due to a Limited Number of Experimental Data

The simulation output distribution obtained in section 3.2 may not be accurate due to simulation model bias in the modeling process. Thus, experimental test data needs to be incorporated to update the output distribution. However, in the presence of epistemic uncertainty due to a limited number of experimental data, many possible candidates of output distributions exist. Among them, a target output distribution against which the simulation model could be validated needs to be selected. In this study, Bayesian analysis is used to quantify many possible candidates of the output distribution. Each candidate output distribution is fitted using Adaptive Kernel Density Estimation (AKDE) as a nonparametric modeling method. The AKDE estimates a PDF using given data without statistical parameters and does not use any parametric PDF [7]. The AKDE adapts local bandwidth, instead of global fixed bandwidth as in the classical KDE, to estimate long-tailed or multi-modal density functions better than the classical KDE. By using ADKE, the uncertainty of the output distribution is quantified by obtaining the posterior distribution of bandwidth $h_n$ given the output test data $y$, using Bayes’ theorem $[2, 8]$. The posterior density of bandwidth $P(h_n | y)$ is
proportional to the likelihood function \( L(y \mid h_0) \) multiplied by the prior distribution \( P(h_0) \) for the bandwidth \( \delta \). A cross-validated log-likelihood function using AKDE function is defined as \([2, 8]\)

\[
\log L(y \mid h_0) = \sum_{i=1}^{n_y} \log f_{-i}(y; h_0) = \frac{1}{(n_y - 1) h(y; h_0)} \sum_{i=1}^{n_y} K \left( \frac{y_i - y}{h(y; h_0)} \right)
\]

where \( n_y \) is the number of output test data, \( y \) is the output test data vector, \( f_{-i}(y; h_0) \) is the leave-one-out AKDE estimator, \( K \) is kernel function, \( h_0 \) is global fixed bandwidth, and \( h(y; h_0) \) is adaptive bandwidth. It is noted that the posterior density of bandwidth cannot be analytically derived. Thus, the Metropolis-Hasting algorithm, a Markov Chain Monte Carlo (MCMC) technique that draws samples from a probability distribution where direct sampling is difficult, has been used to generate a random realization of bandwidth in conjunction with \( P(h_0 \mid y) \). After that, each realization for bandwidth can produce corresponding candidates of output distribution by fitting to the AKDE has given output test data.

The next step is to select a target output distribution from among the candidates as the best approximation of the true output distribution. One may say that the inference ideally should include all posterior distributions. However, considering all candidates of output distributions to come up with one may yield too wide a target output distribution. In this study, a Bayes estimator using the posterior median has been used to select the target output distribution. It is worth noting that the posterior mean and posterior median minimizes Mean Squared Error (MSE) and Mean Absolute Error (MAE), respectively. However, when using MSE, an extreme value like an outlier or noisy data could have a large influence on the result. Thus, it is known that MAE is more robust to outliers that may be safely and effectively ignored in the estimation. Thus, the output distribution at the posterior median is selected as the target output distribution that is the best approximation of the true output distribution.

Once the target output distribution is selected, qualitative model validation can be carried out by directly comparing the similarity between the target output distribution and the simulation output distribution obtained in section 3.2. If the model is an accurate representative of real physics, the simulation output distribution should perfectly match the target output distribution. Furthermore, quantitative model validation can be assessed by measuring the bias of the simulation model, the most probable model prediction error, and the model confidence at a user-specified error level \([9]\).

4. Results

4.1 Identification of Input Distribution Models for an Offshore Jacket Structure Panel

Six widths for corresponding members of the offshore structure panel, shown in Figure 1, are assumed to follow normal distribution because they are geometrical dimensions. The standard deviation (STD) for each is determined based on the manufacturer’s knowledge. The STD of manufacturing tolerance ranges from 0.5 mm to 0.7 mm before post-processing such as sand blasting, grinding, or adding metal powder. After post-processing, the STD of manufacturing tolerance is reduced to the range of 0.05 mm to 0.1 mm. Table 2 lists the statistical properties of the input distribution models for six widths. The coefficients of variation for all widths are in the neighborhood of 4%. For material properties, a limited number (three) of input test data for the yield strength and Young’s modulus is provided, as shown in Table 1. Since only three data are available, a correlation study is not carried out. In addition, systematic behavior is not detected from the given three data sets. Thus, the yield strength and Young’s modulus are treated as independent variables. Figure 2 depicts the plot of the input PDF obtained via the input identification process described in section 3.1. The Weibull distribution has been selected for both the yield strength and Young’s modulus.

<table>
<thead>
<tr>
<th>Label</th>
<th>Type</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Coefficient of variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>t1</td>
<td>Normal</td>
<td>1.8</td>
<td>0.08</td>
<td>4.44%</td>
</tr>
<tr>
<td>t2</td>
<td>Normal</td>
<td>1.3</td>
<td>0.05</td>
<td>3.85%</td>
</tr>
<tr>
<td>t3</td>
<td>Normal</td>
<td>1.2</td>
<td>0.05</td>
<td>4.17%</td>
</tr>
<tr>
<td>t4</td>
<td>Normal</td>
<td>2.5</td>
<td>0.1</td>
<td>4.00%</td>
</tr>
<tr>
<td>t5</td>
<td>Normal</td>
<td>2.5</td>
<td>0.1</td>
<td>4.00%</td>
</tr>
<tr>
<td>t6</td>
<td>Normal</td>
<td>2.5</td>
<td>0.1</td>
<td>4.00%</td>
</tr>
</tbody>
</table>
4. 2 Generation of Dynamic Kriging Model to Obtain Simulation Output Distribution

A total of 166 DOE points—46 initial DOE + 120 sequential DOE (10 DOE at a time)—have been used to generate an accurate DKG model on the variance window. To check the convergence of the DKG model, normalized MSE is evaluated at each iteration and is converged to 2.3E-2. The simulation output distribution is obtained using the DKG model and input distribution identified in section 4.1. The first step is to generate 1M MCS points from the input distribution models. Secondly, the input MCS points are fed to the DKG model to predict the output MCS points. It has been demonstrated that the first four statistical moments (mean, standard deviation, skewness, and kurtosis) of the simulation output distribution using the DKG model show convergence behavior as iterations continue.

4. 3 Target Output Distributions and Quantitative Model Validation Assessment

Since the simulation model could be biased, experimental data is incorporated with the simulation output distribution to approximate the true output distribution. Figure 3(a) depicts the comparison between the simulation and target output distributions and how experimental data is distributed. It is found that the location and shape of the simulation output distribution are quite close to that of the target output distribution. To estimate how biased the simulation model is statistically, the distribution of bias is defined by the convolution of the target output and the simulation output, which can be numerically obtained by Monte Carlo convolution [9]. Like the mean and variance of bias decrease, the simulation model becomes more accurate. When the simulation model is truly representative of real physics, the distribution of bias should become a Dirac delta function. It is also noted that, as the distribution of the biased simulation is spread out on the negative side as shown in Figure 3(b), the simulation model overestimates the true physics. However, an absolute bias comparison may not be appropriate for comparing different models with different performance measures. Thus, model prediction error as a normalized measurement is introduced for advanced quantitative model validation, which is defined by the ratio between the absolute difference between target output and simulation output and target output. A smaller value means the model includes less error. To gain a better understanding of model prediction error, the cumulative distribution function (CDF) of model prediction error is obtained as shown in Figure 3(c).

The obtained CDF of model prediction error can provide (1) the most probable model prediction error and (2) model confidence at the specified error level. First, the model prediction error at the 50th percentile of the vertical axis becomes the most probable model prediction error. It is highlighted that the simulation model of the optimal design of an offshore structure panel is reasonably accurate (5.37% most probable model prediction error). Second, the corresponding confidence value at the user-specified error level is referred to as model confidence level because the CDF value at a specified error level (i.e., the vertical axis in Figure 3(c)) indicates the probability that model prediction is less than or equal to the specified error level. For example, the confidence value is a 99.37% at a 6% error level, which means there is a 99.37% chance that model prediction for an optimized offshore structure panel has less than 6% error when compared with any physical test data under variability. Therefore, it is concluded that the simulation model can be accepted for model usage.
5. Conclusions

Uncertainty quantification and statistical model validation of the optimized design of an offshore structure panel have been successfully carried out. The best-fit input distribution models for two material properties (yield strength and Young’s modulus) have been approximated given a limited number of material testing. Using approximated input models and the ANSYS model, the DKG model has been generated to reduce ANSYS analysis runs for MCS. By incorporating a simulation output distribution with insufficient experimental output data, a target output distribution, which is a good approximation of the true output distribution, has been obtained using a Bayesian inference approach. By comparing the simulation output distribution and the target output distribution, quantitative validation has been carried out. The CDF of model prediction error has been introduced to answer (1) the most probable model prediction error and (2) model confidence at a user-specified error level. It has been demonstrated that model prediction by the simulation model is accurate, as the prediction error is 5.37%. Not only that, model confidence at a 6% error range is 99.37%. Therefore, it has been concluded that the simulation model can be accepted for model usage.

Acknowledgments

The authors wish to acknowledge the technical and financial support of a joint project between the University of Iowa and the Korea Research Institute of Ships and Ocean Engineering (KRISO), as well as technical support from RAMDO Solutions, LLC.

References


COMPARISON OF CHEBYSHEV’S INEQUALITY AND NON-PARAMETRIC B-BASIS TO ESTIMATE FAILURE STRENGTH OF COMPOSITE OPEN HOLE TENSION TESTS

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Abstract

B-basis failure strength represents the lower 10th percentile with a 95% confidence level. In many risk-averse applications the true statistical distribution is unknown, and the B-basis is calculated using a non-parametric formulation. Chebyshev’s inequality makes no assumption about the statistical distribution and can be used to bound the 10th percentile. It is possible to improve these bounds by restricting Chebyshev’s inequality to a class of statistical distributions. B-basis failure strengths are compared by using these methods on a collection of composite open hole tension tests.

Key words: Uncertainty Quantification, B-basis failure strength, Chebyshev’s Inequality, Log-concave CDF

1. Introduction

The failure strength of Open Hole Tension (OHT) tests are an important factor in the design of structures composite for aircraft [1]. B-basis failure strengths are usually considered to conservatively design redundant structures [2]. The B-basis estimates the 10th percentile with a 95% confidence and is usually calculated by assuming a statistical distribution [3]. If the failure strength from OHT samples resembles a known statistical distribution, then estimating the B-basis allowable is a straightforward procedure. Some common distributions for calculating the B-basis include the Normal, Lognormal, and Weibull. Unfortunately, identifying the appropriate distribution can be a difficult task, especially when only a small number of samples are available [4].

Non-parametric statistics can be used to estimate the B-basis calculation when the distribution form is unknown. The MIL-HDBK-17-1F standard [5] has been a popular non-parametric B-basis calculation based on ordered statistics [6]. This method estimates the percentile via a ratio of the lowest observation to some critical observation. Alternatively, it is also possible to bound the tail probability using general Chebyshev’s inequality. In terms of the B-basis, the inequality would represent the worst possible 10th percentile from all possible distributions with a given mean and given variance. The underlying difference between Chebyshev’s inequality is that it attempts to represent the worst possible 10th percentile, while the other B-basis methods attempt to estimate the percentile to 95% confidence.

The bound from Chebyshev’s inequality is anticipated to be overly conservative, but restricting the inequality to a particular class of distributions may produce bounds that are not overly conservative. Grechuk et al. [7] developed a general methodology for deriving Chebyshev’s inequality for a class of distributions. It is possible that Chebyshev’s inequality restricted to a class of distributions may result in bounds that could rival traditional B-basis methods when the true distribution is unknown. The log-concave CDF class used in [8] appears to be a reasonable class, because it includes many distributions typically used in B-basis estimations and has a simple analytical expression.

The paper goes on to describe various methods of estimating the 10th percentile from a small sample. The methods are then applied to the OHT test conducted by [1] on four different configurations. The tests are conducted in batches of three, where each batch consisted of about six OHT tests. The bounded estimates from Chebyshev’s
inequality are compared to the other B-basis estimates.

2. Methods

Composite OHT tests are used to investigate the effect of the hole size on the tensile strength of composite laminates. The OHT tests are performed on a laminate with certain plate width (w) to hole diameter (D) ratios [1-2]. The laminate consists of 20 plies stacked in a sequence of [0/90/0/90/45/-45/90/0/90/0]. The 0 and 90 degree plies each accounted for 40% of the layup, while the 45 degree plies made up the remaining 20%. The methods describe how Chebyshev’s inequality can be used to estimate 10th percentile. Additionally the Normal, Lognormal, and non-parametric B-basis methods are presented.

2.1 Chebyshev’s Inequality

The standard one-sided Chebyshev’s Inequality for estimating the probability that a random variable X lies beyond a threshold a is given as

\[ P[X \leq \mu - a] \leq \frac{\sigma^2}{a^2} \]  

(1)

where \( a > 0 \), \( \mu \) is the true mean, and \( \sigma \) is the true standard deviation. The B-basis for the OHT tests can be estimated by finding the threshold which results in a probability of 10th percentile. This is solved by setting the right hand side of the equation equal to 0, 1, and solving for the \( a \). Chebyshev’s inequality applies to any distribution with both finite mean and finite variance.

Grechuk et al. [7] have developed a general methodology for deriving Chebyshev’s inequality when the random variable X is restricted to belong to a class of statistical distributions, thereby reducing the conservativeness. Symmetric, log-concave, and unimodal are examples of statistical distributions that Chebyshev’s inequality can be restricted to include distributions.

Faridashin et al. [8] used the methodology from [7] to derive the tightest possible Chebyshev’s Inequality for lower tail probabilities when X has log-concave CDF. This is expressed as

\[ P[X \leq \mu - a] \leq \beta \]  

(2)

where \( a > 0 \), \( \beta \in (0,1) \), and is the solution to the equation

\[ \frac{\sqrt{1 + 2\log \beta - \beta}}{\beta - \log \beta - 1} = \frac{a}{\sigma} \]  

(3)

To estimate the threshold for a 10th percentile, set \( \beta = 0.1 \) and solve for \( a \). The log-concave CDF class includes all distributions where the log of the CDF is a concave function. This includes all Normal, Exponential, Gumbel, Laplace, Logistic, Rayleigh, Maxwell, Uniform, Lognormal, and Pareto distributions. Additionally, the log-concave CDF class includes subsets limited to certain parameters of the Weibull, Gamma, Beta, Power Function, Chi-square, and Chi distributions. Figure 1 shows the log of the CDF for the standard normal distribution, and a 4 degrees of freedom t-distribution which does not have a log-concave CDF.

![Figure 1](image)

Figure 1. The standard normal distribution has a log-concave CDF while Student’s t-distribution with 4 degrees of freedom (DOF) does not have a log-concave CDF.

It can be useful to express the threshold in terms of a knockdown factor distance from the standard deviation as
\[ \mu - a = \mu - f \sigma, \]

Using the general Chebyshev’s inequality to estimate the 10th percentile results in a knockdown factor of \( f = 3.0 \), while selecting the log-concave CDF class of distributions results in a knockdown factor of \( f = 1.928 \). The knockdown factor for Chebyshev’s inequality is constant and does not depend upon the number of samples. However, it is based on a given mean and standard deviation, and their accuracy improves with increased number of samples.

2. 2 Normal and Lognormal B-basis

The B-basis bound can be estimated from a Normal distribution using \( \mu - f \sigma \), where \( f \) represents the knockdown factor for a normal distribution. This is calculated according to [9] as

\[ f = \frac{1}{\sqrt{n}} \tau_{1-a/2}\left(1, 1.2816\sqrt{n}\right) \]

where \( n \) is the sample size, \( \tau_{1-a/2}(\delta) \) is the \( (1 - a) \)th quantile of a non-central \( t \)-distribution with \( d \) degrees of freedom and non-centrality parameter \( \delta \). This knockdown factor is also commonly expressed as \( k \), and is tabulated in Table XII of [10]. Bhachu et al. [3] described that the Lognormal B-basis using the same knockdown factor from the Normal distribution as

\[ x_o = \exp(\mu - f \sigma) \]

where \( \mu \) and \( \sigma \) represent the mean and standard deviation of the log transformed sample.

2. 3 Non-parametric B-basis

Bhachu et al. [3] described the non-parametric method for calculating the B-basis value when a collection of samples cannot be modeled by a Normal, Lognormal, or Weibull distribution. The non-parametric B-basis is calculated using ordered statistics, which first requires the sample to be in ascending order as \( x = \{x_1, x_2, \cdots, x_n\} \) for a sample size of \( n \). The following equation is used to calculate the B-basis value

\[ x_s = x_r \left[ \frac{x_1}{x_r} \right] \]

where \( x_1 \) is the lowest observed value in a sample, and \( r \) is the rank index. The parameters \( r \) and \( k \) depend upon the number of samples, and can be looked up in MIL-HDBK-17-1F from [5] Table 8.5.14 when \( n \leq 28 \).

3. Results

Lognormal B-basis, Non-parametric B-basis values, general Chebyshev’s inequality, and log-concave CDF Chebyshev’s inequality are used to estimate the 10th percentile for OHT tests from [1]. Table 1 shows the results from individual batches, which ranged from \( n = 6 \) to \( n = 7 \) samples. The Coefficient of Variation (CV) is the ratio of the mean and standard deviation as \( CV = \sigma/\mu \). The non-parametric B-basis is more conservative than the general Chebyshev’s inequality in 11 of 12 cases. The one case which is less conservative, also happens to be less conservative than the Normal B-basis. The sample set where the Normal B-basis is more conservative than the non-parametric method is \{70, 113, 64, 881, 66, 797, 64, 996, 64, 522, 68, 6, 64, 59\} (ksi), and the non-parametric B-basis is calculated using \( r = 5 \) and \( k = 2.858 \). The knockdown factor from the Lognormal distribution is always smaller than the Normal distribution. The Normal B-basis with six samples is nearly identical to the general Chebyshev’s inequality. Note, however, that the Chebyshev’s inequalities are calculated based on sample mean and standard deviation, which may not be accurate representations of the underlying statistics.

The failure strength from the pooled batches of OHT tests are presented in Table 2. The number of samples here ranged from \( n = 18 \) to \( n = 20 \) samples. Here we see that the most conservative B-basis estimate comes from the general Chebyshev’s inequality, which is about 3 (ksi) smaller than the other B-basis estimates. The B-basis estimates from Normal, Lognormal, non-parametric, and log-concave CDF class are very similar. The knockdown factor of the Non-parametric B-basis ranged from \( f = 1.91 \) to \( f = 2.19 \), while the knockdown factor from the Lognormal distribution ranged from \( f = 1.88 \) to \( f = 1.92 \). With the larger number of samples, the Lognormal B-basis estimate is the least conservative B-basis estimator.

The knockdown factor changes for the Normal, Lognormal, and Non-parametric B-basis estimates depending on the
sample size. A plot of the Normal knockdown factors, and the Chebyshev’s bound is shown in Figure 2. The Chebyshev’s bound crosses the Normal B-basis at 6 samples, while the Log-concave CDF bound crosses at 20 samples. The largest knockdown factor from the non-parametric B-basis method is \( f = 5.99 \) with 6 samples, which is on the order of having only three samples from a Normal distribution.

<table>
<thead>
<tr>
<th>OHT Test Description</th>
<th>( n )</th>
<th>( \mu ) (ksi)</th>
<th>CV%</th>
<th>Normal B-basis</th>
<th>Lognormal B-basis</th>
<th>Non-parametric B-basis</th>
<th>Log-concave CDF Class Bound</th>
<th>Chebyshev’s Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{W}{D} ) = 3 Laminate batch 1</td>
<td>6</td>
<td>57.619</td>
<td>3.461</td>
<td>51.624, 3.006 51.897, 2.870 46.332, 5.660 53.775, 1.928 51.636, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 3 Laminate batch 2</td>
<td>6</td>
<td>63.31</td>
<td>3.408</td>
<td>56.824, 3.006 57.136, 2.862 52.440, 5.038 59.151, 1.928 56.837, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 3 Laminate batch 3</td>
<td>6</td>
<td>57.984</td>
<td>5.218</td>
<td>48.888, 3.006 49.557, 2.785 45.963, 3.973 52.152, 1.928 48.907, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 4 Laminate batch 1</td>
<td>6</td>
<td>63.77</td>
<td>4.033</td>
<td>56.038, 3.006 56.389, 2.870 48.368, 5.989 58.813, 1.928 56.054, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 4 Laminate batch 2</td>
<td>6</td>
<td>65.164</td>
<td>2.762</td>
<td>59.753, 3.006 59.972, 2.885 57.108, 4.476 61.695, 1.928 59.765, 3.000</td>
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<td>( \frac{W}{D} ) = 4 Laminate batch 3</td>
<td>6</td>
<td>60.731</td>
<td>3.411</td>
<td>54.503, 3.006 54.823, 2.852 51.157, 4.621 56.738, 1.928 54.516, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 6 Laminate batch 1</td>
<td>7</td>
<td>66.357</td>
<td>3.362</td>
<td>60.210, 2.753 60.521, 2.616 60.499, 2.626 62.057, 1.928 59.664, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 6 Laminate batch 2</td>
<td>7</td>
<td>68.656</td>
<td>4.828</td>
<td>59.523, 2.755 59.783, 2.677 49.256, 5.833 62.267, 1.928 58.712, 3.000</td>
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<td>( \frac{W}{D} ) = 6 Laminate batch 3</td>
<td>6</td>
<td>64.253</td>
<td>2.527</td>
<td>59.372, 3.006 59.522, 2.914 56.115, 5.012 61.123, 1.928 59.382, 3.000</td>
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<td>( \frac{W}{D} ) = 8 Laminate batch 1</td>
<td>6</td>
<td>67.254</td>
<td>3.202</td>
<td>60.780, 3.006 61.050, 2.881 56.111, 5.174 63.103, 1.928 60.794, 3.000</td>
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<td>( \frac{W}{D} ) = 8 Laminate batch 2</td>
<td>6</td>
<td>70.71</td>
<td>4.913</td>
<td>60.266, 3.006 61.009, 2.792 56.259, 4.160 64.014, 1.928 60.288, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 8 Laminate batch 3</td>
<td>6</td>
<td>69.835</td>
<td>3.769</td>
<td>61.922, 3.006 62.374, 2.835 56.691, 4.994 64.762, 1.928 61.939, 3.000</td>
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<tbody>
<tr>
<td>( \frac{W}{D} ) = 3 All three batches</td>
<td>18</td>
<td>59.638</td>
<td>5.903</td>
<td>52.689, 1.974 53.020, 1.880 52.745, 1.958 52.852, 1.928 49.077, 3.000</td>
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<td>( \frac{W}{D} ) = 4 All three batches</td>
<td>18</td>
<td>63.222</td>
<td>4.414</td>
<td>57.714, 1.974 57.867, 1.919 57.114, 2.189 57.843, 1.928 54.845, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 6 All three batches</td>
<td>20</td>
<td>68.530</td>
<td>4.521</td>
<td>60.874, 1.926 60.936, 1.905 60.916, 1.912 60.870, 1.928 57.720, 3.000</td>
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<tr>
<td>( \frac{W}{D} ) = 8 All three batches</td>
<td>18</td>
<td>69.266</td>
<td>4.386</td>
<td>63.270, 1.974 63.523, 1.891 63.138, 2.017 63.410, 1.928 60.152, 3.000</td>
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</table>
Figure 2. Knockdown factors for the normal B-basis as a function of the number of samples. Histograms of the pooled OHT batches are presented in Figure 3. The pooled batches consist of 18 to 20 samples, and it’s not particularly easy to identify the possible true distribution. This is perhaps most evident with the $\frac{\nu}{D} = 4$ test. It is even more difficult to estimate the distribution from a single batch of just 6 or 7 samples.

![Histograms of OHT batches](image)

- a) All three batches OHT tests $\frac{\nu}{D} = 3$
- b) All three batches OHT tests $\frac{\nu}{D} = 4$
- c) All three batches OHT tests $\frac{\nu}{D} = 6$
- d) All three batches OHT tests $\frac{\nu}{D} = 8$

Figure 3. Histogram of the combined batches of OHT configurations
4. Discussion

There is the potential to improve B-basis estimates with 6 or 7 samples, as there is a large penalty the non-parametric B-basis method over the Normal and Lognormal distributions. In the worst case the knockdown factor of the non-parametric distribution is nearly twice as large as the Normal distribution. Additionally, there is one case where the non-parametric B-basis is not as conservative as the Normal distribution.

Differences between the B-basis methods are hardly noticeable with samples totaling 18 to 20. The Normal, Lognormal, non-parametric, and Log-concave CDF bounds are basically identical, with the Lognormal resulting in the least conservative estimate. While the non-parametric B-basis could have a huge knockdown factor of $f = 5.99$ with 6 samples, the knockdown factor reduced to approximately $f = 2.0$ with 18 samples, which is on a par to the other mentioned methods.

One potential issue is that Chebyshev’s inequality requires the true mean and true standard deviation, while bounded estimates are performed on the sample mean and standard deviation. It is possible to account for the potential uncertainty in the sample mean and standard deviation with Bootstrapping, which wouldn’t make additional assumptions about the potential true distribution. This would result in a distribution of bounded estimates from the possible means and standard deviations. The estimated bound could then be selected for an appropriate confidence level. It is worthwhile to point out that in this case where the uncertainty in the sample means and standard deviations are neglected, the general Chebyshev and Log-concave CDF bounds are comparable to the other B-basis calculations.

5. Conclusion

Traditional B-basis methods attempt to estimate the 10th percentile to a 95% confidence level for a collection of samples. If the true distribution is unknown, then non-parametric statistics are used to estimate the B-basis. The non-parametric estimate will be very conservative for a small number of samples, as a cost of not knowing the true distribution. For instance, there is an OHT batch of 6 samples where the non-parametric B-basis is 6 standard deviations away from the mean. There may be better ways to approach non-knowing the true distribution with six or 7 samples than the non-parametric B-basis. An additional concern is that the non-parametric B-basis are not always more conservative than the B-basis from a Normal distribution. The general Chebyshev’s inequality can be applied to estimate a bounded 10th percentile for any distribution with a known mean and variance. With 6 or 7 samples in the OHT tests, the general Chebyshev’s inequality is comparable to the Normal distribution. Chebyshev’s inequality restricted to the log-concave CDF class is comparable to a Normal B-basis with about 20 samples. Although with 20 samples, B-basis estimates from the Normal, Lognormal, and non-parametric methods are nearly the same. It’s possible to restrict Chebyshev’s inequality to only apply to a particular class of statistical distributions. The log-concave CDF class applies to many distributions that are commonly used to estimate tail probabilities. Perhaps some restricted class (like the Log-concave CDF) could be used to improve B-basis estimates for unknown distributions with 6 or so samples.

References

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UNCERTAINTY PROPAGATION USING L-MOMENTS WITH SCARCE SAMPLES INCLUDING EXTREMES

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Abstract

Inevitable uncertainties in input variables often lead to uncertainty of system output. Treatment of these uncertainties requires propagating them through the system model. Often characterization of uncertainties that are to be propagated itself is challenging. When only limited samples are available, researchers assume a distribution and use the data to estimate the parameters alone. This requires knowing the distribution a priori which is not the case in engineering applications. Inaccurate assumptions about the distribution would lead to flawed decisions. In addition, extreme values of the limited data are usually presumed to be outliers and neglected which leads to the wrong estimation of the parameters. Therefore, it is desirable to have a method that can account for the extremes yet be insensitive to the parameter estimation. We propose using L-moments which are known to be less sensitive to extremes, to characterize the randomness and propagate that through the system model. Propagating higher order L-moments allows for better characterization of the output distribution but requires an additional computational cost. The accuracy of output distribution obtained using L-moments are found to be less sensitive to the extremes or extremes. The efficacy of the proposed approach is demonstrated using a test function and an engineering example.

Key words: Uncertainty Quantification, Classical Moments, L-Moments, Extremes

1. Introduction

In structural design, uncertainties are inevitable and considered [1-2] as the root cause of poor product performance. These uncertainties typically occur in material properties, manufacturing parameters and externally applied forces. Description of the uncertainties associated with the input variables is not usually available readily. The distributions and their parameters, if quantified based on scarce samples lead to large errors, which when propagated through the models result in amplified errors in the output space. In addition, the scarce samples might also include extremes, which will only worsen the errors. Instead, the uncertainties with the extremes can be directly propagated and the quantities of interest such as the mean and deviation of the output response, quantified. The classical moments are more sensitive to extremes present in the data but L-moments are less sensitive and suffer less from sampling variability compared with the classical moment [3-6]. The first four L-moments are defined as in Eq. (1).

\[
\begin{align*}
\lambda_1 &= E[X_{1,1}] ; \\
\lambda_2 &= \frac{1}{2} E[X_{2,2} - X_{1,2}] ; \\
\lambda_3 &= \frac{1}{3} E[X_{3,3} - 2X_{2,3} + X_{1,3}] ; \\
\lambda_4 &= \frac{1}{4} E[X_{4,4} - 3X_{3,4} + 3X_{2,4} - X_{1,4}] \end{align*}
\]

(1)

Where \(X_{j,m}\) is the \(j\)th smallest variable in a sample of size \(m\) and \(E\) stands for expectation. These moments give measures of location and scale parameter. Generally, designers assume the output distribution follows a normal distribution even for a nonlinear function.

In this study, the performance of L-moments is tested with a test function and an engineering example for limited samples with the inclusion of extremes. L-moments and the classical moments are used to characterise the output distribution and the output distribution obtained from limited samples with the inclusion of extreme are compared with 10^6 samples.
2. Methodology

The hypothesis of this work is that L-moments are robust compared to the classical moments, from an estimation perspective and similarly it gives a better characterization of the output compared to the classical moments. We are interested in investigating this for small sample size, especially in the presence of extremes. We adopt the following investigation strategy to test the hypothesis. For limited samples, the outputs are computed and compared with the actual population obtained from $10^6$ samples. Extreme values are included in the limited samples and the corresponding output is compared with the population. The difference is used as the comparison metric and shown in Eq. (2).

$$\text{Difference}, \ D = (.)_{\text{Estimated}} - (.)_{\text{Exact}}$$

Where $(.)_{\text{Estimated}}$ is moments estimated from samples and $(.)_{\text{Exact}}$ is moments estimated from $10^6$ samples. Chi-square test[7] is used to identify the type of distribution of the response and a significance level of the chi-square test is fixed as 5%. The distributions Normal, Lognormal, Logistic, Gumbel, Generalized extreme value, Generalised Pareto and Gamma are used to fit the response data. Distributions obtained from scarce samples are compared with the distribution obtained from $10^6$ samples by Kolmogorov-Smirnov test (KS test)[8].

3. Test Problem

A function, $y=x^3+x^2$ is used to test the hypothesis of L-moments and to adopt the methodology mentioned above. ‘$y$’ is a response and ‘$x$’ is a input variable. The input variable follows a normal distribution with mean, 6 and standard deviation, 0.6. Figure 1 shows the probability density function of the input variable. The actual distribution of the response is obtained using the outputs generated using $10^6$ input samples. In this study, sample sizes 30, 60, 90 and 120 are used. Maximum or minimum values from $10^6$ samples are considered as extreme and the same is added to the limited samples. These samples are generated for 100 times and the corresponding Classical(C) and L-moments are computed. Figures 2 and 3 shows the comparison of standard deviation and second L-moment ($l_2$) obtained from scarce samples with extreme to the $10^6$ samples. It can be clearly seen that L-moments are less sensitive and it is less affected by the sampling variability compared to the classical moments in the presence of extremes in samples. Similarly, comparison of skewness and kurtosis with the L-skewness and L-Kurtosis are follow the same pattern. For example, the median of the standard deviation obtained from 30 samples with extreme is 119, 74 which is 63, 37% away from the median of the actual standard deviation obtained from $10^6$ samples. Likewise, the median of $l_2$ in the presence of the extreme is 56, 06 which is 36, 99% away from the actual median of $l_2$. These values clearly indicate that L-moments are less affected by the inclusion of extremes. In the same way, higher order moments, skewness ($\gamma$), kurtosis ($\beta$) are compared with the L-skewness ($\tau_3$) and L-kurtosis ($\tau_4$) for the output response and obtained similar observations Figure 4. Shows the comparison of C and L-moments based on the error metric Difference, D and it reveals that L-moments are less affected by the sampling variability. For sample sizes 60, 90 and 120 same studies was repeated and it was found that L-moments work better compared to the classical moments. Since the hypothesis of L-moment is satisfied, it can be used to characterise the output response distribution.

![Figure 1. Input distribution ~ N(6,0,6)](image-url)
output response data with p-value, 0, 3949 and the other distributions used in this study are rejected the null hypothesis in the chi-square test. The parameter values of the gamma distribution for the population are shape parameter 12, 48 and scale parameter 20, 73. L-moments are also obtained and the corresponding values are $l_1$, 258, 83 and $l_2$, 40, 92 and the gamma distribution parameters based on L-moments are shape parameter, 12, 47 and scale parameter 20, 74. From the limited samples data, parameters of the gamma distribution are obtained based on both C and L-moments. Figure 6 gives the comparison of the sample distributions to the actual distribution. KS test is conducted between the sample distribution and the actual distribution and between sample distribution and sample with extreme distribution.

![Input variable](image)

(a) Standard deviation
(b) Second L-moment

Figure 2. Comparison of standard deviation and second L-moment ($l_2$) for the input variable

![Output response](image)

(a) Standard deviation
(b) Second L-moment

Figure 3. Comparison of standard deviation and second L-moment ($l_2$) for the output response

![Output response](image)

(a) $\sigma \& l_2$
(b) $\gamma \& \tau_3$
(c) $\beta \& \tau_3$

Figure 4. Comparison of classical and L-moments for the output response
4. Engineering Example Flat-rolling Metalworking Process Used to Manufacture Metal Plates

Flat-rolling process is illustrated in the Figure 7 and the aim of this process is to minimize the number of passes required to achieve final plate thickness[9]. The maximum change in thickness attainable in a single pass $\Delta H_{\text{max}}$ is given in Eq. (3).

$$\Delta H_{\text{max}} = \mu f R$$  \hspace{1cm} (3)

Where $\mu f$ is the friction coefficient $R$ is the radius of rolls, 40, 6 cm.

Figure 7. Flat-rolling metalworking process
The statistical properties of the friction coefficient are shown in Table 1. Friction coefficient distribution is obtained using the Pearson system[10] and shown in Figure 8. Where ‘h’ is the height of the plate, ‘V’ is the velocity of the plate, ‘0’ is denoting initial process and ‘f’ is denoting final process.

Table 1. Statistical Properties of the Friction Coefficient

<table>
<thead>
<tr>
<th>Statistical property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.35</td>
</tr>
<tr>
<td>Variance</td>
<td>$9 \times 10^{-4}$</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.7</td>
</tr>
<tr>
<td>Excess kurtosis</td>
<td>0.2</td>
</tr>
</tbody>
</table>

In this study, the sample size is considered to be 30 and extremes are the maximum or a minimum of $10^4$ samples. Pearson system is used to obtain the distribution for the $\Delta H_{max}$, based on four C-moments and GEV distribution is used for L-moments. Scare samples are generated from input distribution and the corresponding response is computed. This process is repeated for 100 times. Figure 9 shows the comparison of standard deviation and second L-moment ($l_2$) obtained from scarce samples with extreme to the $10^5$ samples. It is observed that L-moment hypothesis is satisfied and Figure 10 shows the comparison of C and L-moments based on the error metric Difference. D. It is seen that L-moments are less sensitive to sampling variability. Comparison of the sample distributions with the actual distribution shown in Figure 11. KS test is used to compare the distributions and the results clearly indicate that distribution obtained from the L-moments are close to the actual distribution.

Figure 9. Comparison of standard deviation and second L-moment ($l_2$) for the output response
5. Conclusion

12-moments are proposed to characterise the output response distributions with scarce data since it is less affected by the sampling variability and less sensitive to the extremes present in limited data. A comparative study has been conducted between the classical moments and L-moments and it is found that L-moments work better, in the presence of extremes. This is evaluated by the error metric. Difference. In this study, two examples are considered and the observations from that are clearly indicates that distributions obtained using L-moments are better than the classical moments for limited samples.

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REVISITING ENSEMBLE OF SURROGATES USING MACHINE LEARNING ALGORITHMS

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Abstract

Surrogate models have become a mainstay in tackling engineering applications where a large number of function evaluations is necessary but are computationally expensive. A surrogate model is a cheaper alternative to the computationally expensive response estimation. In the event of multiple surrogates, ensemble approaches have been successfully used for robust approximations and to learn information about the design space such as the possibility of high error. However, there are multiple ways in which the ensemble itself can be organized. Developments in machine learning algorithms, which emerge from a classifier perspective, are widely being adopted for regression as well and have been demonstrated to perform well for large data. However, the availability of literature that addresses the efficacy of these approaches with the small sample is scarce. In this work, we create ensembles of both classical surrogates such as polynomial response surface, Kriging and machine learning algorithms such as Lasso, random forest etc. We also compare two different weighting schemes. We investigate the performance of different models using three analytical functions. We observe that cross validation method performs better than splitting the data into training and validation set and proceeding to build the model. We also find that the ensemble models consistently perform well compared to any individual surrogate model.

Key words: Surrogate Modeling, Machine Learning, Ensemble

1. Introduction

To understand or analyze or optimize a system, the knowledge of the output of the system for different kinds of input conditions is required. The procurement of these output values of a system is often an expensive process, especially in the case of engineering applications. Surrogate modeling aids in this process by providing a cheaper approximation model built by using limited output evaluations. Researchers compared different surrogate models for robustness, efficiency, scalability and simplicity. Many studies also investigated the suitability of certain surrogates for specific applications. Most of these studies discovered that no single surrogate model performed the best for all problems. At this juncture, the ensemble of surrogates was introduced as a viable alternate [1]. Goel et al. showed that ensemble can be used to identify regions of large uncertainties and a weighted average model lends a better robust approximation than any individual surrogate. They also propose different ways in which the weights can be computed such as weighting scheme based on the relative magnitude of global errors and another scheme which controls the tradeoff between the importance of average and the importance of individual surrogate. In [2], Erdem et al. discuss the computation of a weight factors based on the prediction variance of the models and correlation between residual errors of the models among other schemes and propose a scheme that finds the weights by minimizing the ensemble error at the validation points. Similar discussions on different weighting schemes are presented in [3].

With the advent of Machine Learning (ML), several new techniques have emerged for surrogate modeling. The idea of the ensemble was borrowed from the machine learning community. However, there is scant literature on ensemble models constructed using ML based surrogate models for scarce data. This forms the motivation for our work. We study the performance of 13 different types of surrogate modeling approaches which include the classical approaches
and the ML algorithms. In addition, we compare two approaches of constructing the ensembles discussed in [1] and [2]. Three analytical functions namely: Branin-Hoo, Camelback and Hartman3 have been used to perform the study. The rest of the paper is organized as follows; section 2 provides a brief overview of different models that are considered in this work. In section 3, ensemble methods are discussed. Section 4 presents results followed by a summary of observations in section 5.

2. Surrogate Models

The idea about surrogate modeling is to construct an approximation $\hat{f}(\mathbf{x})$ for a function $f(\mathbf{x})$ given a set of $n$ observations $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ derived from the function at a set of input values $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$. Here $\mathbf{x}$ is the input vector such that $\mathbf{x}^{(i)} = (x_{i1}, x_{i2}, \ldots, x_{ip})$ and $p$ is the number of input dimensions. Models described in subsections 2.1 to 2.4 can be categorized as linear regression based models. While the models in subsections 2.5 to 2.8 are also built on linear regression, they have an additional regularization component. Methods discussed in subsections 2.9 and 2.10 are decision tree based models.

2.1 Linear Regression

Linear regression is one of the earliest methods developed and widely used because linear models are simple to construct and offer great interpretability such as: effect of input on the output and the effect of interaction between the inputs on the output. In linear regression, the model is linear in terms of coefficients $\beta$ that need to be found and the mathematical representation of a linear regression model is as shown in Eq. The input vector $x$ need not be linear and moreover different transformations on the inputs are utilized to improve the scope of the approach, these are known as basis methods. The simplest form of such transformations is the polynomial basis where $x = (1, x_1, x_2, \ldots, x_p)$. This representation can be extended to $p$ dimensions,

$$\hat{f}(\mathbf{x}) = \mathbf{x}^T \beta$$

There are different methods to find the unknown coefficients but the most popular method employed is the Ordinary Least Squares (OLS). In OLS, $\beta$ is obtained as a solution to the minimization of the sum of the squares of residual errors,

$$\min_\beta \sum_{i=1}^n (y^{(i)} - \hat{f}(\mathbf{x}^{(i)}))^2$$

2.2 Radial Basis Function (RBF)

Increasing non-linearity in the model using a polynomial basis becomes a complex task. It involves choosing the order of the model, estimating coefficients and discarding unnecessary terms of the polynomial [4]. Expressing the linear model in terms of radial basis functions avoids such complexity. The function approximation then can be written as

$$\hat{f}(\mathbf{x}) = \mathbf{w}^T \boldsymbol{\psi} = \sum_{i=1}^n w_i \phi(\| \mathbf{x} - \mathbf{c}^{(i)} \|)$$

Where $\mathbf{c}^{(i)}$ is the $i$th basis function center and $\boldsymbol{\psi}$ is the vector containing the values of the basis functions at the test point from the centers of the basis. Gaussian, multi-quadratic and inverse multi-quadratic are some of the commonly used radial basis functions. The expression for the Gaussian radial basis function is given in Eq. (4).

$$\phi(r) = e^{-r^2/\sigma^2}$$

The additional parameter $\sigma$ in the basis function is a hyper-parameter in this model which can be an arbitrary fixed value or can be obtained by searching in the hyper-parameter space for the value that minimizes the error in the model.

2.3 Kriging Surrogate Model

Kriging surrogate considers the output values $Y(\mathbf{x}^{(i)})$ as a realization from a random process that has zero mean and a non-zero covariance. The output values are considered to be correlated with each other in the following way.

$$\text{cor}[Y(\mathbf{x}^{(i)}), Y(\mathbf{x}^{(i)})] = \exp\left(-\sum_{j=1}^n \theta_j |x_j^{(i)} - x_j^{(i)}|^p\right)$$

23
The basis function used for the correlation in Eq. (5) is similar to the Gaussian basis (Eq. (4)) used in RBF, but the difference is \( \theta \) and \( \rho \) are distinct for each dimension and \( \rho \in [1, 2] \). These are the hyper-parameters in Kriging surrogate which can be obtained by tuning methods such as grid search or as a result of an optimization problem.

2.4 Multivariate Adaptive Regression Spline (MARS)

MARS makes use of piecewise linear splines and their products as basis functions in the linear regression model as shown in Eq. (6) where \( h_w \) is a product of piecewise linear splines. The art of the approach is building a good set of basis \( h_w \). The coefficients for the model are obtained via least squares fit. A sample piecewise linear spline is shown in Eq. (7).

\[
\hat{f}(x) = \sum_{w=0}^{M} \beta_w h_w
\]

\[(x-t)_+ = \begin{cases} x-t, & \text{if } x > t, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad (t-x)_+ = \begin{cases} t-x, & \text{if } x < t, \\ 0, & \text{otherwise}. \end{cases} \]

The python implementation of this method is done by using py-earth package. Hence we call the model built from it as Earth model in this work. We also consider Light Earth model in our work. Light Earth is based on FAST MARS. Readers interested in further details about the MARS approach are referred to [5].

2.5 Ridge Regression

Ridge regression is a type of regularization on the linear regression model where, in addition to the residual sum of squares, the \( L_2 \) norm of the coefficients is also minimized. There is a correlation between the inputs in the linear regression model. OLS performs poorly and the coefficients exhibit a high variance. This kind of regularization imposes a penalty on the size of the coefficients. Larger the value of \( \alpha \), the more the coefficients get penalized. The idea is to avoid correlated inputs from having large coefficient values.

\[
\min_{\beta} \| x^T \beta - y \|^2 + \alpha \| \beta \|^2
\]

2.6 Least Absolute Shrinkage and Selection Operator (LASSO)

LASSO uses the \( L_1 \) norm of the coefficients to regularize. The difference between LASSO and ridge regression is the usage of the \( L_1 \) norm leads to zero values for coefficients of some of the input terms. Hence, LASSO can be used as a variable selector for some applications.

\[
\min_{\beta} \frac{1}{2n} \| x^T \beta - y \|^2 + \alpha \| \beta \|_1
\]

2.7 Elastic Net

In Elastic net, the regularization is performed with both \( L_1 \) and \( L_2 \) norm of the coefficients. Elastic net also can be used as a variable selector.

\[
\min_{\beta} \frac{1}{2n} \| x^T \beta - y \|^2 + \alpha_0 \| \beta \|_1 + \frac{\alpha (1-p)}{2} \| \beta \|_2^2
\]

2.8 Support Vector Machine (SVM)

Support Vector Machine for regression(SVR) is a particular implementation of SVM. In SVR, the aim is to find the approximation \( \hat{y}(x) \) that has at most a deviation of magnitude \( \varepsilon \) from each of the training data. The errors are considered zero as long as they are less than \( \varepsilon \). Consider the linear regression model in Eq. (1), in SVM the coefficients \( \beta \) are obtained by minimizing

\[
H(\beta) = \sum_{i=1}^{n} V(y_i - \hat{y}(x_i)) + \frac{\lambda}{2} \| \beta \|^2
\]

Where

\[
V_\varepsilon(r) = \begin{cases} 0, & \text{if } |r| < \varepsilon, \\ |r| - \varepsilon, & \text{otherwise} \end{cases}
\]

2.9 Random Forest

Random forest shows an example of a bagging type ensemble technique. Bagging or bootstrap aggregation is used for reducing the variation of the estimated prediction function by building many independent models and combining them
by weighted average (regression) or majority vote (classification). The key difference in the random forest model is that it builds a large collection of de-correlated trees and averages them[6]. The many models \( T_s(x) \) is built on different bootstrap samples or random sub-samples are obtained from the data and the final prediction model is obtained as

\[
\hat{y}_v^\theta(x) = \frac{1}{B} \sum_{s=1}^B T_s(x)
\]  

(13)

2. 10 Gradient Boosting Machinery (GBM)

Boosting is another type of ensemble technique where the models are not built independently but sequentially. In the boosting method, the observations with the highest error are carried forward. The gradient of the loss function is utilized in finding the iterative model. Other efficient implementations of GBM known as light GBM and XGBoost are also considered in our work. For further details regarding these approaches, the reader is referred to[7]. For further details on different ML approaches, the reader is referred to [4] and [6] for classical surrogate modeling approaches.

3. Ensemble of Surrogates

Methods discussed in subsections 2.9 and 2.10 are ensemble methods where the ensemble model is a combination of a simple predictor model like linear regression. However, when we use the word ensemble the model here, it is not restricted to using one type of model, instead we use all the models that were discussed in section 2 to build the ensemble models. The weight of a model depends on the relative measure of errors among the models considered. The errors used for computation of the weights can be local or global. In our study, we test the effectiveness of two approaches based on global errors proposed in the literature. We denote the models as EM1 and EM2, where the weights for EM1 and EM2 are calculated using the approach followed in [1] and [2], respectively, which is discussed in detail below.

3.1 Error Metrics

A surrogate model needs to be validated before it is used for prediction or any other intended application. In order to validate a surrogate model, additional observations are required which is a luxury that is often unaffordable. Hence, the set of observations are divided into a training set and validation set prior to building the model. This approach is called the validation approach and the Root Mean Square Error (RMSE) at these validation points is used to assess the model accuracy. The general idea is that the accuracy of the surrogate models increases with the increase of size of the training set but this adversely affects the validation set and this tradeoff is sometimes undesirable. Cross-validation provides an alternative at the expense of computational time. Given a data set \( D = \{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(n)}, y^{(n)})\} \), we extract \( D_s \) the set without the \( i^{th} \) observation; \( D = \{(x^{(1)}, y^{(1)}), \ldots, (x^{(i-1)}, y^{(i-1)}), (x^{(i+1)}, y^{(i+1)}), \ldots, (x^{(n)}, y^{(n)})\} \). The surrogate model is built on this data set and error at the removed point is calculated. repeating this process for all the observations we obtain errors at all the observations. The final surrogate model is built using all the observations and the RMSE of the predicted errors (PrESS RMSE) serves as the validation metric. The type of cross-validation approach that is chosen here is Leave-One-Out Cross-Validation (LOOCV). There are other variations in cross-validation approach depending on how many points are removed, popularly known as k-fold cross-validation.

3.2 Goel et al Approach (EM1)

The weights of each surrogate \( w_i \) are obtained as

\[
\begin{align*}
\omega_i^* &= (E_i + aE_{\text{avg}})^\beta w_i = \frac{\omega_i^*}{\sum_{i} \omega_i^*} \\
E_{\text{avg}} &= \frac{\sum_i E_i}{M}, \quad a = 0, 0.5, \beta = -1.
\end{align*}
\]  

(14)

Where \( E_i \) is the RMSE or PrESS RMSE error of each surrogate model computed at the validation points, \( M = 13 \) is the total number of surrogate models considered in our work.
3. 3 Acar et al Approach (EM2)

\[
\min_{\psi} E_{\text{one}} \\
\text{s.t. } \sum_{i=1}^{M} w_i = 1
\]

(15)

Where

\[
E_{\text{one}} = \sqrt{\frac{1}{v} \sum_{i=1}^{v} e_{\text{one}}^2} \\
e_{\text{one}} = \sum_{i=1}^{M} w_i e_i
\]

(16)

Here \( e_{\text{one}} \) is the point-wise error of the ensemble model computed as the weighted sum of the errors from each surrogate model. The weights \( w_i \) is found by minimizing the root mean square of the ensemble errors at \( v \) validation points with a constraint such that the sum of the weights is unity.

4. Methodology & Results

Latin Hypercube Sampling (LHS) was used to create the sample points in the input space and the function expressions were used to generate responses at these sample points. To make the observations and inferences independent of the sample set chosen, we create 1000 instances of design of experiments for all the test problems. Numerical settings used for each problem are given in Table 1. The number of test points is computed by considering 21 points along each dimension. The entries for the order of polynomial are used in linear regression and the values of spread are used in RBF.

<table>
<thead>
<tr>
<th>Table 1. Numerical Setup for the Test Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Variables</td>
</tr>
<tr>
<td>No. of Design Points</td>
</tr>
<tr>
<td>No. of Test Points</td>
</tr>
<tr>
<td>Order of Polynomial</td>
</tr>
<tr>
<td>Spread</td>
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<tr>
<td></td>
</tr>
</tbody>
</table>

Our objective in this study is to compare the two ways of splitting the data for model building and validation. We call cross-validation based approach as the PrRESS approach and the other approach as validation approach. In the PrRESS approach, the PrRESS RMSE computed at validation points is used to compute the weights of the ensemble models whereas the validation approach uses the RMSE at validation points. In each approach, we compare the accuracy of the 13 surrogate models as well as two ensemble models. We also investigate whether the validation error is reflective of the actual error in the model. For this purpose, we compare the RMSE/PrRESS RMSE at validation points with RMSE at the test points. We normalize the error metrics by the range of the responses at the validation or test points in order to be able to compare the error values across the example problems.

In the PrRESS approach, the number of validation points would be the same as the number of training points (12-Branin-Hoo, 20-Camelback and 40-Hartmann3). In validation approach, we consider about one-third of the training points as validation points (4-Branin-Hoo, 6-Camelback and 12-Hartmann3) and these points are chosen such that they are representative of the sample space.

4. 1 Test Functions

1. Branin-Hoo function

\[
f(x) = \left( x_2 - \frac{5.1 x_1^2}{4\pi^2} + \frac{5 x_1}{\pi} - 6 \right)^2 + 10 \left( 1 - \frac{1}{8\pi} \right) \cos(x_1) + 10 \\
x_1 \in [-\pi, \pi], x_2 \in [0, 15]
\]

(17)
Figure 1. Branin-Hoo function: Normalized error metric. a. PrESS approach-validation errors and b. Validation approach-validation errors. c. PrESS approach-test errors and d. Validation approach-test errors.

Figure 2. Branin-Hoo function: Comparison plot of quartiles of normalized error metric. a. PrESS approach (test errors vs. validation errors). b. Validation approach (test errors vs. validation errors).
2. Camelback Function

\[ f(x) = \left( 4 - 2.1x_1^2 + \frac{x_1^4}{3} \right)x_1^3 + x_1x_2 + \left( -4 + 4x_2^2 \right)x_2^2 \]

\[ x_1 \in [-3, 3], x_2 \in [-2, 2] \]

Figure 3. Camelback function: Normalized error metric, a. PrESS Approach-validation errors and b. Validation approach-validation errors c. PrESS approach-test errors and d. validation approach-test errors

Figure 4. Camelback function: Comparison plot of quartiles of normalized error metric a. PrESS approach (test errors vs. validation errors) b. Validation approach (test errors vs. validation errors)
3. Hartman 3

\[ f(x) = -\sum_{i=1}^{c} \exp \left( -\sum_{j=1}^{3} \alpha_{ij} (x_j - p_{ij})^2 \right) x_i \in [0,1] \]

\[ c = (1, 0.1, 2, 3, 0.3, 2)^T \]

\[ A = \begin{bmatrix} 3, 0 & 10 & 30 \\ 0, 1 & 10 & 35 \\ 3, 0 & 10 & 30 \\ 0, 1 & 10 & 35 \end{bmatrix} \]

\[ P = 10^{-5} \begin{bmatrix} 3689 & 1170 & 2673 \\ 4699 & 4387 & 7470 \\ 1091 & 8732 & 5547 \\ 381 & 5743 & 8828 \end{bmatrix} \] (19)

Figure 5. Hartman3 function: Normalized error metric. a. PrESS approach-validation errors and b. Validation approach-validation errors c. PrESS approach-test errors and d. Validation approach-test errors.

Figure 6. Hartman 3 function: Comparison plot of quartiles of normalized error metric a. PrESS approach (test errors vs. validation errors) b. Validation approach (test errors vs. validation errors)
5. Summary of Observations

The boxplots in Figures 1, 3 and 5 show that the relative performance of the models is the same across PrESS approach and validation approach. These trends remain the same even across the example problems considered with the exception of linear regression. We observe that the PrESS approach is superior to the validation approach both in terms of the median error and also the variance in error. A simple explanation could be that more points are used to build the surrogate models in the PrESS approach than the validation approach but this need to be further investigated.

The quartile plots in Figures 2, 4 and 6 show the extent to which the validation error from a model captures the actual error in the model. It can be observed that, for all example problems considered, ensemble approaches especially EM2 validation errors are reflective of the actual errors in the model. In terms of whether validation error is reflective of the actual errors, the models built based on the PrESS approach to perform well.

Among the individual surrogate models, Kriging and RBF consistently perform better. Linear regression underperforms in the case of camelback and hartman3 even though the order of the polynomial used is higher than the other regression based models like a ridge, lasso and elastic net. EM2 can assign zero weights to some models so it can serve the purpose of the model selection metric.

References

7. G. Ke et al., LightGBM; A Highly Efficient Gradient Boosting Decision Tree.
OPTIMIZATION DESIGN OF MTMD FOR RADIAL GATES

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Abstract

For solving the abnormal operation and failure of radial gates due to flow-induced vibration, an effective design approach for the use of Multiple Tuned Mass Dampers (MTMD) in radial gates is proposed in this study. First, the typical forced vibration equations of radial gates with MTMD are established on the basis of the generalized radial gates-MTMD system model and assumptions, then the conventional selection strategy of the natural frequencies of MTMD is improved. Second, in combination with the parametric vibration of radial gates with MTMD, the derivation of the objective function of structural minimum vibration response considering both the vibration patterns is obtained, and complete optimization is achieved through MATLAB-ANSYS joint simulation. To evaluate the validity of the method, a practical project is adopted as a numerical study, and the evaluation results show that the MTMD designed using this method can significantly reduce structural responses when subjected to flow excitation. Moreover, the parametric vibration can also be effectively suppressed.

Key words: Radial Gate, Multiple Tuned Mass Damper, Parameter Optimization, Flow-Induced Vibration, Vibration Control

1. Introduction

Radial gates are one of the most common features used in hydraulic engineering; the gate stability is directly related to the safety of the entire reservoir. However, the mechanism of dynamic instability of radial gates has not yet been proven, which leads to the lack of design process, and present specifications are not sufficient to meet the ever-changing construction needs. In recent years, many engineering projects have shown that radial gates show varying levels of dynamic response according to the water flow action during the operation of dynamic water control gates or partial gate openings. Under some special coupled or waste vibration states, significant radial gate vibration takes place, and it may cause dynamic instability in the arm, even leading to damages in the entire system [1]. Therefore, the vibration control of radial gates should be resolved. Based on the structural characteristics and forces acting on radial gates, a previous study focused on the vibration reduction in the arm [2]. Meanwhile, according to forms of radial gate vibration response excited due to the cavity flow, Karman vortices, extreme excitation, and so on, the vibration patterns are categorized into forced, self-excited, and parametric vibrations. Among these patterns, self-excited and parametric vibrations are generally caused by harmonic excitation, which is a type of forced vibration. However, the direction of excitation and response is orthogonal in parametric vibration; it is different from other vibration forms and not investigated in the previous studies on vibration control.

The design of the main frame of radial gates is different from the conventional civil structure system. The joint between the arm and the main beam needs to bear the bending moment, shear force, and axial force. If the vibration isolation technology is adopted, the integral rigidity decreases; therefore, other energy dissipation technologies are considered to reduce the vibration without affecting the integral rigidity. Moreover, parametric vibration analysis results show that energy dissipation dampers such as viscous and friction dampers have limitations with regard to this nonlinear vibration behaviour. However, Tuned Mass Dampers (TMD) can adjust and control the structural dynamic characteristics by means of energy transfer, wherein the natural frequency is targeted instead of the damper ratio of the structural system. In terms of post-repair maintenance, the conventional structure requires the re-reinforcement of the plastic region, whereas TMDs only require inspection or replacement. Moreover, the seismic load is a typical pulse load, whereas the flow-induced vibration has a long duration, and the lagging of damping control has a negligible effect on TMDs.
With the development of TMDs, the Multiple Tuned Mass Damper (MTMD) is used in the design stage because it offers effectively improved the stability, narrow band, and the poor achieveability in the application of traditional Single Tuned Mass Dampers (STMDs). Therefore, in this research, based on the modelling and analysis of forced and parametric vibrations of radial gates with MTMD, a new optimized design method for the system is established.

2. Radial Gate-MTMD System Model with Forced Vibration

2.1 Analytical Formulation

The radial gate is represented by its mode-generalized system in the specific vibration mode being controlled using the model reduced-order method. Each TMD is also modelled as an SDOF system, which is set to have different dynamic characteristics. Figure 1 shows the radial gate model and assumptions. Meanwhile, a number of new parameters are introduced, \( \xi_r \), \( \xi_i \), and \( \xi_t \), represent the damping ratio of radial gates, \( r \)-th TMD and average, respectively. \( r \) and \( \lambda \) represent the frequency ratio of the \( r \)-th TMD and an excitation, respectively. \( m_i \) and \( \mu_i \) represent the mass ratio of the \( r \)-th TMD and MTMD, respectively, which can be defined as:

\[
\xi_r = \frac{c_r}{2m_0\omega_r}, \quad \xi_i = \frac{c_i}{2m_0\omega_i}, \quad \xi_t = \frac{\Sigma c_i}{n}, \quad r = \frac{\omega_r}{\omega_0}, \quad \lambda = \frac{\omega_r}{\omega_0}, \quad m_i = m_i/m, \quad \mu = \frac{\Sigma m_i}{m}.
\]  

(1)

Figure 1. Radial gates-MTMD system model and assumptions

Assumptions:

1. Dynamic response of radial gates is in the elastic range.
2. The additional MTMD has little effect on the original vibration mode of the main structure.
3. According to the modal analysis, the maximum displacement response occurs at the 1/2 or 1/4 length of the radial arm. Thus, MTMDs are installed in this position, as shown in Figure 1 (b).

The equation of motion for this system is:

\[
M \ddot{z} + C \dot{z} + K z = (H) p(t)
\]

(2)

where \( M, C \) and \( K \) are the mass, damping and stiffness matrices respectively, \( (H) \) is the direction matrix related to the Flow excitation force \( p(t) \).

The matrices, \( M, C \) and \( K \) are expressed as:

\[
M = \text{diag}[m_0, m_0, m_0, \cdots, m_n]
\]

\[
C = \begin{bmatrix} C_0 + \sum C_d & -C_d \\ [-C_d]^T & \text{diag}[C_d] \end{bmatrix} \quad \text{where} \quad C_d = [C_1, C_2, \cdots, C_n]
\]

\[
K = \begin{bmatrix} K_0 + \sum K_d & -K_d \\ [-K_d]^T & \text{diag}[K_d] \end{bmatrix} \quad \text{where} \quad K_d = [K_1, K_1, \cdots, K_n]
\]

(3)

The state-space representation of Eq. (2) is as follows:

\[
\dot{z} = Az + Bp(t)
\]

(4)

where \( \{z\} \) is the state vector \( \{x, \dot{x}\}^T \) and

\[
A = \begin{bmatrix} 0 & I \\ -M^{-1}C & -M^{-1}K \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}\{H\} \end{bmatrix}
\]

(5)

by substituting steady-state responses \( p(t) = e^{i\omega t} \) into Eq. (4), the transfer functions of the structure with the MTMD can be formulated as:

\[
H(\omega) = (i\omega I - A)^{-1}B
\]

(6)
Then the displacement dynamic magnification factor of the structural response is derived as:

$$ R_s = \frac{\text{Re}^2(\lambda) + \text{Im}^2(\lambda)}{\sqrt{\text{Re}^2(\lambda) + \text{Im}^2(\lambda)}} \quad (7) $$

where

$$ \text{Re}(\lambda) = 1 + \sum_{i=1}^{n} \mu \frac{\xi^2 (r_i^2 - \lambda^2) + 4 \lambda^2 \xi r_i^2}{(r_i^2 - \lambda^2)^2 + 4 \lambda^2 \xi r_i^2} \quad \text{Im}(\lambda) = -\left[ 2 \xi + \sum_{i=1}^{n} \frac{2 \mu \lambda^2 \xi r_i^2}{(r_i^2 - \lambda^2)^2 + 4 \lambda^2 \xi r_i^2} \right] \quad (8) $$

2. 2 Effect of the Natural Frequencies Distribution of MTMD

In [3], it is assumed that the stiffness and damping coefficients of each TMD in the MTMD are the same, and the natural frequencies of the MTMD are uniformly distributed over a small range in the neighbourhood of the structural natural frequency. In this work, the value ranges of the ith natural frequency $\omega_i$ are the same as those in [3]; however, the distribution is not uniform. It is obtained on the basis of the criterion that the minimization of the maximum value of the displacement dynamic magnification factor $\min_{\omega_0} \max_R \xi$ and $\sum_{\omega_0/n=\omega_0}$. Comparing the optimal dynamic magnification factor $R_{\omega}^{\text{opt}}$ proposed in this study to the results obtained in [3], it is observed that $R_{\omega}^{\text{opt}}$ is lower, as indicated in Figure 2, which implies that the MTMD design can suppress excessive vibration significantly, thereby offering improved robustness.

<table>
<thead>
<tr>
<th>Table 1. Comparison with Other Result Research</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li (2009)</td>
</tr>
<tr>
<td>n</td>
</tr>
<tr>
<td>$k_i$</td>
</tr>
<tr>
<td>$c_i$</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
<tr>
<td>$\xi$</td>
</tr>
<tr>
<td>$\mu^{\text{opt}}$</td>
</tr>
<tr>
<td>$\beta^{\text{opt}}$</td>
</tr>
<tr>
<td>$\xi_r^{\text{opt}}$</td>
</tr>
<tr>
<td>$R_{\omega}^{\text{opt}}$</td>
</tr>
</tbody>
</table>

![Figure 2. Optimal dynamic amplificatory coefficient-excitation frequency ratio curves](image)
3. Radial Arm-MTMD System Model with Parametric Vibration

The parametric vibration, a unique form of gate nonlinear vibration, appears in radial arms. In the 1980s, according to the experimental phenomena, Zhang [4] recognized parametric vibration was one of the reasons that caused dynamic instability of arm, and then presented a simple analytical method, afterward plenty of research about this area came out from both domestic and aboard.

![Figure 3. Radial arms vibration model](image)

Here, based on characteristics of radial gates, taking the end moment caused by water head and the geometric nonlinear caused by large displacement into consideration, according to the Figure 3, the nonlinear dynamic equilibrium equation of the arm with ith MTMD can be written in:

\[
\begin{align*}
\left\{ \begin{array}{l}
\ddot{V} + 2\xi \omega_0 \dot{V} + (\omega_0^2 + \omega_d^2) V + \omega^2 + d_\alpha u_d + e + c_\alpha (\dot{V} - \dot{V}_c) + k_\alpha (V - V_c) = 0 \\
\dot{m}_\alpha \ddot{V} + c_\alpha (V - \dot{V}) + k_\alpha (V - V) = 0
\end{array} \right.
\tag{9}
\end{align*}
\]

where

\[
2\xi = \frac{\mu}{m}, \quad \omega_0 = \omega^i (1 + \theta), \quad \omega_d^2 = \frac{\pi^2 H_i}{mL^2} + \frac{E_i \pi^i}{mL^2}
\]

\[
\begin{align*}
\theta &= \left[ \left( -\frac{M_i \pi^2 L_i^2}{4EI} \left( \frac{L_i}{4} + \frac{1}{3} + \frac{1}{2\pi^2} - E \right) \right] H_i + \frac{4M_i A_i^i}{EI} \left( \frac{M_i L_i^2}{4EI} - 1 \right) \right] \left( \frac{-mL^2}{2} \right)

a &= \left[ \left( -\frac{M_i \pi^2 L_i^2}{4EI} \left( \frac{L_i}{4} + \frac{1}{3} + \frac{1}{2\pi^2} - E \right) \right] H_i + \frac{4M_i A_i^i}{EI} \left( \frac{M_i L_i^2}{4EI} - 1 \right) \right] \left( \frac{-2EI \pi^i}{L^2} \right) \left( \frac{-mL^2}{2} \right)

b &= \left[ \left( -\frac{M_i \pi^2 L_i^2}{4EI} \left( \frac{L_i}{4} + \frac{1}{3} + \frac{1}{2\pi^2} - E \right) \right] H_i + \frac{4M_i A_i^i}{EI} \left( \frac{M_i L_i^2}{4EI} - 1 \right) \right] \left( \frac{-2EI \pi^i}{L^2} \right) \left( \frac{-mL^2}{2} \right)

c &= \left[ \left( -\frac{M_i \pi^2 L_i^2}{4EI} \left( \frac{L_i}{4} + \frac{1}{3} + \frac{1}{2\pi^2} - E \right) \right] H_i + \frac{4M_i A_i^i}{EI} \left( \frac{M_i L_i^2}{4EI} - 1 \right) \right] \left( \frac{-2EI \pi^i}{L^2} \right) \left( \frac{-mL^2}{2} \right)

c_\alpha &= \frac{4c_\alpha}{\pi m} \sin \left( \frac{\pi \xi \alpha}{L} \right), \quad d_\alpha = \left[ \frac{2AM_i}{\pi I} \left( \frac{M_i L_i^2}{8EI} - 1 \right) - m \frac{L_i^2}{\pi} \right] \left( \frac{-mL^2}{2} \right)

e &= \left( \frac{ML_i^2}{8EI} - 1 \right) \left( \frac{-mL^2}{2} \right)
\end{align*}
\]

In which \(m, c, k\), and \(k\) are the mass, damping and stiffness of ith MTMD respectively, \(V\) and \(V_c\) are the midspan displacement of arms and ith MTMD displacement along \(y\)-axis, respectively. \(L\) is the length of the arm, \(m\) is mass density, \(A\) is the cross-sectional area, \(EI\) is the flexural rigidity, \(M_i\) is assumed to be the initial end moment, \(H\) is the axial hydrostatic pressure of the arm, \(\xi\) is the arm damping ratio, \(\omega_0\) is the first-order natural frequency of the arm considering the comprehensive effect, \(\omega_d\) is the excitation frequency, \(A\) is the excitation amplitude, \(x_{cl}\) is the arrangement of MTMD along \(x\) direction.
4. Optimal Parameters of the MTMD and Numerical Study

4.1 Optimal Model

The objective function is used to evaluate the effectiveness of MTMD installed in radial gates under flow excitation. It consists of the maximum dimensionless displacement dynamic magnification factor $R/D_o$ and arm displacement $V/D_a$, the $\alpha$ and $\beta$ represent scale factor.

The design variables are the MTMD parameters, i.e., mass ratio, frequency and damping ratio of each TMD, based on a large number of research results, reasonable values ranges of them should be considered. Therefore, the optimization problem can be presented as follows:

Find: $\mu$, $\alpha$, $\xi$

Minimize: $\text{Max } f(\mu, \alpha, \xi) = \text{Max } \{\alpha (R/D_o) + \beta (V/D_a)\}$

Subjected to: $\sum \alpha_i/n = \alpha_i$, $\mu < 5\%$, 0.1 $\leq \beta \leq$ 0.5, $n = 7$, 0.00 $\leq \xi \leq$ 0.10

During the optimization process, the main control model of the structure is initially obtained by using ANSYS and then the optimization criterion equations used MATLAB to determine the optimal parameters of MTMD. Moreover, in practical engineering, the feedback adjustment of the design parameters is required to be carried out by using ANSYS according to the measured values obtained under varying conditions of operating water head, flow state, and gate opening. Therefore, by using MATLAB-ANSYS joint simulation, a new MTMD optimization design method for nonlinear flow-induced vibration is developed.

4.2 Numerical Study

A numerical study is conducted to examine and analyze the efficiency of the presented technique of optimization strategy. The beam and arm of radial gates adopt I section and box section, respectively. Other dimension parameters are shown in Figure 4. Its material properties are as follows: linear density of 386 kg/m and Young's modulus 210 GPa.$^{[5]}$

![Figure 4. Main frame of radial gates](image)

The Optimal Single TMD Parameters Based on the Proposed Design Approach Given in Table 2

<table>
<thead>
<tr>
<th>$\mu^{opt}$</th>
<th>$\beta^{opt}$</th>
<th>$\xi^{opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0167</td>
<td>0.1979</td>
<td>0.0504</td>
</tr>
</tbody>
</table>

The flow excitation loading designed by using the ideal power spectrum depicted in $[5]$ is applied to the main frame of the radial gate. The displacement response and the parametric resonance before and after installation of the MTMD are shown in Figures 5 and 6, respectively. The effectiveness of MTMD is proven and a reduction of 46-72% in vibration control is attained.
5. Conclusions

In the previous analyses of the mechanical properties of radial gates, reducing structural stiffness or using energy dissipation dampers was found to be ineffective for vibration control. Owing to this, the MTMD is selected to reduce its vibration. Further, a new MTMD optimization design method for nonlinear flow-induced vibration is proposed; it involves the optimization of forced and parametric vibration and is developed by using MATLAB-ANSYS joint simulation. Finally, from the analysis of the performance before and after the installation of MTMD under flow excitation loading and parametric vibration, the proposed optimized design can be recommended for the efficient vibration control in radial gates.

Acknowledgments

The authors are grateful for the support from National Natural Science Foundation of China under Grant Nos. 51179164 and 51478354.

References


OPTIMISATION OF DIFFUSION DRIVEN DEGRADATION PROCESSES

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Abstract

It is known that nature always strives for optimal processes in order to prevent early damage or failure, for example bones which regulate the tissue in high stress areas or the optimal shape of trees. Many materials in nature are highly complex as they are designed for long duration. Taking the nature itself as a role model means taking its complexity into account, especially when considering materials which suffer from chemical attacks and the degradation of materials. From this point of view, we create an algorithm which considers degradation processes triggered by chemical substances which depend on diffusion conditions in a finite element (FE) framework. In addition, we perform design optimisation with stress restrictions by taking evolutionary processes as a role model.

For structural analysis, the continuum mechanical theory will be used to describe the coupled aspects between mechanics, diffusion and mass resorption. With a thermodynamically consistent formulation of a single-phase open system we will evaluate the constitutive equations for the stresses, mass source and the evolution of growth. The diffusion is a gradient-based formulation of nutrient motion in analogy to Fick’s law. Thus, chemical reactions of specific nutrients trigger degradation and growth processes leading to large deformations. The outlined continuum mechanical theory will be embedded in an FE-framework. An enhanced kinematic concept allows the decomposition of growth and elastic deformations and, furthermore, enables the separation of growth deformation, geometry and physics. Gradient information resulting from the sensitivity analysis will be applied for the gradient based mathematical optimisation.

As a representative example of the application of the theory, we will discuss a simple illustrative structure affected by chemical concentration. However, the task is the computation of the interaction between diffusion, degradation and mechanical stresses. With an optimisation algorithm of the design, we can evaluate the optimal geometry of a structural component which ensures stability despite destructive influences of degradation of mass.

Key words: Structural Optimisation, Coupled Problems, Growth Optimisation, FEM

1. Introduction

This paper focuses on the general outline of the algorithm combining a specific structural coupled problem to an optimisation framework. In detail, we deal with the coupling of diffusion and growth/degradation processes embedded in a design optimisation algorithm. The advantages of the power of a Matlab allow the application of a Matlab optimisation toolbox within a finite element method (FEM) formulation. The material formulation is implemented in Fortran by using MEX-file interfaces.

In the following, we present the computational model. On the one hand, we introduce the material description of a coupled diffusion and growth/degradation process and the structural analysis by applying the FEM. We give a short overview of the kinematic assumption and the evaluation of the most important equations. On the other hand, we outline the connection to an optimisation algorithm referring to a simple example.
2. Structural Analysis

We present a coupled diffusion and growth/degradation approach with a permeable structure allowing a gradient-based flow of concentrations. The continuum describes a one-phase approach with continuum modelling of a growth/degradation process. In analogy to growth in a living structure we refer to a volumetric growth approach. We apply a multiplicative decomposition of the deformation gradient into an elastic part \( F^e \times \) and a growth part \( F^g \) \([2-3.5]\)

\[
F_X = F^e_X F^g_X
\]

(1)

2.1 Kinematics

The kinematic concept can be seen in Figure 1. We introduce the coordinates in the reference configuration \( x \), in the actual configuration \( \tilde{x} \) in the growth space \( \tilde{X} \) and in the parameter space \( \theta \), namely the convective coordinates. Furthermore, we apply to map between the configurations for geometry \( \tilde{x} \), for deformation \( x \) and for growth \( \tilde{X} \). Partial derivatives of the mapping with respect to the local coordinates \( \theta \) result in the convective tangent vectors \( G \) \( \cdot \) \( g \) and \( G \). The gradients of the mapping between the parameter space and the introduced configurations lead to the local, fundamental gradients \( K \), \( F \) and \( G \). These gradients are important two-point tensors providing a transformation between the configurations. The commonly used deformation gradient \( F_X \) is related to the reference and actual configuration. Thus, Lagrangian mapping from the reference configuration to the actual configuration \( \tilde{X} \) can be described by the composition \( \tilde{x} = \tilde{X} \circ \tilde{X}^{-1} \). The deformation gradient \( F_X \) is multiplicatively split into the elastic deformation gradient \( F^e_X \) and growth deformation gradient \( F^g_X \). This kinematic concept enables the description of all tangent operations in the continuum mechanical concept. For further information, see \([1]\).

![Figure 1. Kinematic concept](image)

Assuming a mass source/sink term \( R \) in the reference configuration leads to the kinematic evaluation of growth with an update of the initial density \( \rho_0 \) to the resulting density \( \rho \) in the reference configuration

\[
\rho_t = \rho_0 + \int_0^t R \, dt
\]

(2)

where \( t \) denotes time. The density \( \rho = \tilde{\rho}_g = \rho \) is assumed to stay constant. Following Lubarda and Hoger \([4]\), we assume isotropic growth with an isotropic stretch ratio \( \nu \). We create the growth deformation gradient \( F^g_X \) and the determinant of the gradient \( J^g \) as follows

\[
F^g_X = \nu I_X, \quad J^g = \nu^3
\]

(3)

(4)

Based on the introduced fundamental gradients and regarding a constant density \( \rho_0 \), we evaluate the following kinematic restriction for the stretch ration
\[ y = \frac{p_0}{\rho_0} \]  

2. 2 Constitutive Framework

The second law of thermodynamics implies that an isolated system’s entropy never decreases. For the evaluation of the stresses, the degradation process and the diffusion of the concentration, we consider the constitutive theory and regard the entropy inequality, by following the thermodynamically consistent evaluation. Introducing the Helmholtz free energy \( \Psi_X \), together with the assumption that the released entropy never decreases in an isolated system, leads to the basic equation for a thermodynamically consistent evaluation, the so-called Clausius-Duhem inequality

\[-\rho_0 (\Psi_X)' + P_X : F_X' - R_c (\Psi_X + \frac{1}{2} \mathbf{x}'_c \cdot \mathbf{x}'_c) \geq 0 \]  

(6)

Wherein the notation (\( \cdot \)\)’ indicates the material time derivative and \( P_X \) is the first Piola-Kirchhoff tensor. With the introduced two-point gradients we can transform this tensor with

\[ P_X = F_X (F_X')^{-1} S_X (F_X')^{-T}. \]  

(7)

The evaluation of the entropy inequality results in the following sets of constitutive equations Eq. (8)-(10). The second Piola-Kirchhoff type stresses tensor \( S_X \) in the growth configuration as follows

\[ S_X = \frac{\rho_0}{\rho_0} [\lambda 6 \sqrt{\text{det} C_X} (C_X)^{-1} - \mu (C_X)^{-1}] + \mu \lambda \mathbf{x}'_c. \]  

(8)

wherein \( C_X \) is the elastic right Cauchy Green deformation tensor and where \( \lambda \) and \( \mu \) are the material parameters.

We combine the introduced kinematic assumption for growth Eq. (2) with a constitutive evaluation of material degradation/accumulation. Thus, a constitutive evaluation of the mass source/sink term \( R_c \) in Eq. (9) provides the connection between both approaches. Assuming a coupling between the chemical impact of concentrations to the degradation or accumulation of material leads to the following constitutive law

\[ R_c = M_c c_j. \]  

(9)

The velocity of the concentration \( x'_c \), Eq. (10), is calculated in analogy to Fick’s law, assuming that the velocity moves from regions of high concentrations to regions of low concentrations

\[ x'_c = D_c \text{grad} c_j. \]  

(10)

The concentrations are described with \( c_j \), diffusion parameter \( D_c \) considers the permeability of the continuum, and \( M_c \) is the molar masses of the concentrations.

2. 3 Numerical Solution

The previously introduced material formulation for the coupled diffusion and growth/degradation model leads to a reduction of the unknown quantities and to a set of coupled differential equations. A numerical approximation, the FEM is applied for further procedures. In order to close the system of partial differential equations the unknown quantities are solved by the choice of governing equations. This approach is used to solve the set of degrees of freedom, including the displacement and the concentration evaluated with the balance of momentum and the balance of mass for concentrations. This set of unknown quantities is determined via the Galerkin procedure where the balance equations are transformed to their weak form and weighted by the independent test functions \( \mathbf{\bar{x}} \) and \( \mathbf{\bar{x}}_c \). The relevant weak formulations in the reference configuration are introduced as follows

\[ \int_{\Omega_0} P_X : \text{Grad} \mathbf{\bar{x}} dV = \int_{\Omega_0} t_X \cdot \mathbf{\bar{x}} dA \]  

(11)

\[ \int_{\Omega} (c_j \mathbf{\bar{x}}_c - x'_c \cdot \text{Grad} \mathbf{\bar{x}}_c) dV = \int_{\Omega} x'_c \cdot \mathbf{\bar{n}} c_j dA \]  

(12)

In order to solve the time-dependent non-linear problem, we apply the implicit Newmark scheme assuming constant acceleration. With quadratic ansatz functions for the FEM approach, we reached a stable numerical evaluation.

3. Structural Optimisation

The main focus of this paper is the illustration of a coupled problem embedded in a structural optimisation framework. For the sake of simplicity, we apply a Matlab optimisation toolbox as a solver. This combination allows full access to
the power of a Matlab, thus using the introduced FE-model. With the information regarding the objective function, the side conditions and the gradients of both we can generate an optimal system.

The optimisation problem as follows

\[
\min J(s) \quad \text{subject to:} \quad A \cdot s \leq b \quad \text{linear constraint} \\
\begin{array}{l}
\quad s' \leq s \leq s'' \quad \text{bounds for design}
\end{array}
\]

(13)

Therein, \( s \) is the design and \( J(s) \) is the objective function. The linear constraint is evaluated from a linearised form of the von Mises stress \( \sigma_{VM} \), following

\[
A = \frac{\partial \sigma_{VM}(X(s))}{\partial s} \quad \text{and} \quad b = \sigma_{VM}^{\max} - \sigma_{VM}^{\text{ind}}.
\]

(14)

A general outline is plotted in Figure 2. The entire computation runs in a Matlab framework, which includes a connection to the open software gmsh for mesh creation on the one hand, and an interface to a Fortran code using MEX-file interfaces on the other. Starting point of the algorithm is the mesh creation depending on the free choice of key points and mesh size. With the automatic connection to gmsh, an FE-mesh is evaluated. In this example we focus on a structure with a hole, depending on four key points as printed in Figure 2. The structural analysis computes the coupled diffusion and growth/degradation model with MEX-file interfaces. For the optimisation of the initial problem, an objective function, design parameters and side conditions are required. In this example, the objective function \( J(s) \) is the area of the structure aiming a reduction of material using the parameters of the hole as the design parameters \( (s_1, s_2) \), as shown in Figure 3.

\[
J(s_1, s_2) = (ab) - (s_1, s_2 \pi)
\]

(15)

![Figure 2. Optimisation algorithm](image)

![Figure 3. FE design](image)

Fulfilling the side condition for the stress restriction leads to the following condition for a maximum value of the von Mises stress \( \sigma_{VM}^{\max} \).

\[
\sigma_{VM} \leq \sigma_{VM}^{\max}
\]

(16)

Satisfying the tolerance for the design parameters \([s_1, s_2]\) ensures the basic geometry condition with an upper bound \([s_1, s_2]_{\max}\) and a lower bound \([s_1, s_2]_{\min}\)

\[
[s_1, s_2]_{\min} \leq [s_1, s_2] \leq [s_1, s_2]_{\max}
\]

(17)

With this information, the gradients of the objective function and side condition, we apply a Matlab optimisation toolbox as a solver, creating new design parameters.

4. Example

In this paper we discuss the coupled effects of diffusion leading to the degradation of material and to stress. The illustrative example is performed on a structure with a hole. The boundary problem is shown in Figure 4. We consider a concentration inflow from the left side to the outflow on the right side of the structure, where in the inflow condition increases in dependence of the time. The computation runs over 50 time steps with a time step size of 0.1
and with the given parameters in Table 1. A stable mechanical environment is enabled by fixed displacements, as shown in Figure 4.

Table 1. Material Properties

<table>
<thead>
<tr>
<th>Material Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus $E$</td>
<td>10 GPa</td>
</tr>
<tr>
<td>Poisson’s Ratio $\nu$</td>
<td>0.2</td>
</tr>
<tr>
<td>Initial Density $\rho_0$</td>
<td>3000 $\text{kgm}^{-3}$</td>
</tr>
<tr>
<td>Molar Mass $R_e$</td>
<td>8.043 $\text{kgmol}^{-1}$</td>
</tr>
<tr>
<td>Diffusion parameter $D_f$</td>
<td>0.1 $\text{m}^2\text{s}^{-1}$</td>
</tr>
</tbody>
</table>

![Figure 4. Boundary conditions](image)

![Figure 5. Diffusion of a concentration after 50 time steps without degradation of material. The gradient of the concentration increases in each time step, ending in a maximum difference of 13 $\text{nmol} \text{L}^{-1}$ at the inflow and 0 $\text{nmol} \text{L}^{-1}$ at the outflow.](image)

First we evaluate the diffusion of the concentration with the applied concentration gradient via the boundary condition in Figure 5. The speed of the concentration can be influenced by the diffusion coefficient and concentration gradient. The black streamlines show the flow of the concentration, surrounding the hole and the concentration motion from the left to the right side of the structure.

In the next step, we focus on the degradation of material triggered by the concentration, leading to a reduction of area. As the concentration motion depends on the diffusion through the continuum, we can observe a spatial distribution of the impacted area. The main material reduction takes place close to the inflow area where the highest concentration is located. Thus, the impact on the von Mises stress takes place in this area with the maximum stress close to the hole as printed in Figure 6. In Figure 7, the impact on the determinant of the growth tensor $J_e$ is shown, evaluated in the point with the maximum stress impact.

![Figure 6. Evaluation of the von Mises stress after 50 time steps. With a maximum stress of 1200 GPa](image)

![Figure 7. Evaluation of the determinant of the growth gradient $J_e$ in the marked point within 50 time steps. With a minimum determinant of 0.78.](image)
Furthermore, this destructive process will be optimised with a geometry optimisation. The central task is a structural design optimisation within an FE-framework. The side conditions restrict the von Mises stresses which play a major role in the decreasing process. The design variables allow a change of the hole parameters, leading to the new design as printed in Figure 8. We apply a Matlab optimisation toolbox as a solver. The minimisation of the area, with a von Mises stress restriction of 1200 GPa, depends on the motion of the concentration. In this example, the concentration plays a supporting role for material minimisation, so that the degradation process is exhausted until the stress restriction is reached and the minimum of the objective function is calculated. The optimisation leads to a material reduction of 17.38%.

![Figure 8. Optimised design](image)

5. Conclusion

A general algorithm combining the structural analysis of a diffusion and growth/degradation problem with an enhanced kinematic concept has been embedded in an optimisation framework. The example shows the reasonable analytical response of a structure affected by diffusive concentrations which release degradation. Furthermore, the numerical optimisation illustrates that the algorithm is able to generate an optimised design which depends on the development over time, the spatial conditions and the diffusion parameters. Further investigations will focus on a variational approach for sensitivity information. The introduced kinematic concept allows a separation of design, material quantities and growth framework. This basis allows the computation of gradient information of the structural response simultaneously to the sensitivity response in an FE-framework.

References

A TOPOLOGY OPTIMIZATION BASED DESIGN MODEL FOR ENHANCED ENERGY ABSORPTION AND FAIL-SAFE BEHAVIOUR

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Abstract

Density based Topology Optimization (TO) as used in commercial applications and problems of scale is largely restricted to static and linear problems. Also, plastic behaviour and failure of the material are often disregarded in the design optimization approach. Resulting optimized structures are typically statically determinate and lack in robustness in regard to energy absorption and graceful degradation in overloading, misuse or extreme events. Thus, local material failure can lead to the abrupt total collapse of the whole structure.

A topology optimization based highly automated design model is proposed for enhancing energy absorption and fail-safe behaviour of optimized structures. For this purpose a local damage model is used in a Multi-Model topology optimization (MMO) approach for generation of redundant load paths. A safe measure is introduced that is parametrically optimized based on characteristic values of energy absorption and fail-safe behaviour of different topologies in a nonlinear explicit analysis.

The design model is developed with a high degree of automation, such that effective weighting of technical design criteria can be achieved and design optimization results can be directly manufactured (e.g. by 3d printing) with little manual CAD. The application of the approach is demonstrated in simulation analysis as well as experimental testing of 3d printed structures.

Key words: Multi-model Topology Optimization, Fail-Safe Design, Energy Absorption, Nonlinear Material, 3d Printing

1. Introduction

In this paper, a topology optimization (TO) based design model is presented that aims for generating robust and safe lightweight design solutions for loaded functional parts. An extensive overview of TO methods is given in [1]. The most common TO approach Solid Isotropic Material with Penalization (SIMP), as described in [2], which is widely implemented in commercial software and already applied to problems of scale is largely restricted to static and linear problems. The most common optimization approach for lightweight design applications is to minimize the compliance of the structure with a volume constraint. This is a very effective way to optimize the stiffness of a structure, but in the classical approach it completely disregards local stresses. Stress constraint TO is a topic of ongoing research. So far, it is not available to full extend and for complex engineering problems in commercial software. That is due to the high nonlinearity of stress-constrained problems, the local nature of stress concentrations as well as computational cost [3]. Global stress measures that are implemented in some commercial codes, e.g. Optistruct do not gain substantially better designs, in terms of stress concentration prevention [4]. Thus, topology optimized structures can have significant stress concentrations, e.g. due to notch effects that can lead to local material failure. Also, optimized structures can lack in robustness in regard to uncertainties or energy absorption and graceful degradation in overloading, misuse or extreme events. Local material failure can lead to the abrupt total collapse of the whole structure.

The terms “robust” and “safe” are widely used in engineering fields with a variety of characteristics and definitions. In structural optimization related publications, robust structures are defined as less sensitive to uncertainties [5]. On this account, different topology optimization methods have been developed. Deterministic methods approximate the level of uncertainty, e.g. by extreme load cases or worst case material defects. The robust optimal design problem for topology optimization is then often a minimax problem, also referred to as worst case topology optimization. It is numerically hard to solve and delicate to the quality of the worst case approximation, see [6]. Stochastic methods are
classified as Robust TO (RTO) and Reliability-based TO (RBTO) methods, see e.g. [5]. Derived safety factors can be significantly smaller than in a classical worst case formulation, but are still a very conservative choice, as stated in [7].

Apart from the fields of structural optimization or system theory, the robustness of structures was also defined as the ability to withstand local failure without disproportionate damage and progressive collapse, see [8]. This definition can also be directly related to the term safety and the engineering practice of defining safety factors for structures. In the field of topology optimization, new methods have been described recently, namely Fail-Safe Topology Optimization (FSO) and Redundant Topology Optimization (RTO), see e.g. [9-11], that can be collated to before mentioned understandings of robustness and safety. Still, nonlinear material behaviour as well as transient mechanics are disregarded in those approaches as well as the well-developed RTO and RBTO approaches to robustness. The implementation of nonlinear material and nonlinear geometry in topology optimization methods has also been addressed in the past, e.g. [12], but has not found its way into commercial software, yet. Also, optimized structures are typically statically determinate. Nonlinear and transient response was addressed by methods like equivalent static loads method (ESLM) or hybrid cellular automata (HCA) aiming for example for crashworthiness designs, see [13]. Still, robustness and safety related design constraints like fail-safe or redundancy have not been addressed with those methods.

Apart from the multitude of methods and approaches to robustness and safety of topology optimized structures, it still remains an underdeveloped issue in the field of topology optimization, especially in the regime of industrial applications and commercial applicable software tools. Here a design model is presented, that is to a large extent implemented on basis of commercial available software, takes nonlinear material behaviour in to account and is also able to include transient mechanics and failure behaviour. The approach makes use of different available methods, namely SIMP, FSO and Multi-model optimization (MMO) and aims at providing design engineers with an automated design model for performant but safe design solutions. A new safety measure for optimized structures is introduced and maximized in the optimization. It is based on the volume specific deformation energy and energy absorption in the plastic deformation and failure regime of the deformation. Also, a graceful, non-progressive collapse of the structure in a worst case scenario is integrated. In doing so, the safety and robustness of the structure can be designed with an additional focus on the nonlinear behaviour and safety measure in the worst case, apart from solely optimizing for stiffness. Consequently, no or smaller extra safety factors need to be added. It is expected, that in doing so the total safety factor can be reduced and lightweight parts can be designed more effectively. The approach is implemented in 2D and highly automated incorporating commercial design tools. The potential of the approach is shown in an example problem and validated in 3D printing of optimized structures and mechanical testing.

2. Design Model for Enhanced Energy Absorption and Fail-safe Behaviour

Overview of Procedure and Interactions

In Figure 1 an overview of the design model, included procedures and interactions is given with illustrations of a standard cantilever beam min. compliance problem.

![Figure 1. Overview of design model and procedures.](image)

The model consists of a standard min. compliance SIMP topology optimization and subsequent nonlinear design optimization, with the goal of enhancing the energy absorption and fail-safe behaviour of the topology result. In
contrast to the FSO approach discussed in [9-10], no random local damage model is used, but instead the damage models for the MMO are derived from a SIMP topology result and density distribution. The MMO has then the goal of finding alternative load paths resulting in a structural redundancy in the resulting topology result. The MMO models are weighted with a characteristic parameter, which is the design variable for the nonlinear optimization. The objective is to maximize a defined safety measure function, which is calculated from a nonlinear analysis of the MMO topology result. The nonlinear model for the analysis is automatically generated from the MMO density distribution by a density threshold and smoothing operation. The whole design model is automated to a high degree. As yet, the critical manual and empirical step is the derivation of the models for the MMO, as described in the next section, as well as the correlation of boundary conditions between the initial model and the nonlinear model.

3. Implementation for an Example Design Problem

The implementation is described on the structural problem of a tension beam supported on one side and loaded by a unidirectional and uniform distributed tension force. The material is a 3d printing aluminum alloy named Scalmalloy with assumed isotropic mechanical properties. The following material characteristic mechanical values are taken from in house tensile test measurements with not heat or surface treated 3d printed specimen: $E=59$ GPa, $R_y=210$ MPa, $Rm=319$ MPa, $\rho=2.7$ g/cm$^3$. The design problem and standard SIMP topology result solved in the commercial software Altair Optistruct is illustrated in the upper area (0) of Figure 2.

3.1 Topology Optimization with Redundancy of Load Paths in the Multi-model Approach

From the density result (0) in Figure 2, where black regions represent dense material and white regions are empty, three areas are derived for the MMO model definition. The support (I) and load application area (III) are equally defined as non-design space for both derived models (1) and (2) from Figure 2. Additionally, areas (II) are filtered empirically for the definition of a local damage model. For this purpose, the Elastic modulus is varied and the filter areas (II) share the same Elastic modulus in both derived models.

![Figure 2. Generation of models for MMO from density result of min. Compliance optimization.](image)

The min. compliance objective for the MMO is defined as the weighted compliance of the two models suspected to a volume constraint, as in Eqs. (1-3):

$$\min_{\rho} \bar{\epsilon} (\rho) = a \bar{\epsilon}_1 (\rho) + c_1 (\rho)$$

(1)

$$V^* - \sum_{e=1}^{N} V_e \rho_e \geq 0$$

(2)

$$0 < \rho_{\min} \leq \rho_e \leq 1$$

(3)

In Eq. (1) $\bar{\epsilon}$ is the compliance of the resulting structure, whereas $c_1$ is the compliance of model (1) and $c_e$ the compliance of model (2). The models are weighted with the characteristic parameter $a$, which is to be optimized in the nonlinear optimization. In Eqs. (1-3) $\rho$ is the density function or element density, respectively. $V^*$ is the volume constraint and $V_e$ the volume of element $e$. 

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The simultaneous minimization of the compliances of both models with local damage areas, when suitably determined, leads to the formation of redundant load paths in the resulting design. The shared areas of both models are needed to traverse the damage regions where the Elastic modulus is set to a minimum, so that a consistent model with adequate transition regions in between support and load application non-design areas is defined.

![Topology results of the MMO in dependency of the characteristic parameter $a$.](image)

In Figure 3 MMO results for the described configuration from Figure 2, are illustrated. For values of $a$ below 0.5 and more significantly below 0.1 the second model has higher weighting and the density result converges to a result similar to the SIMP result of the initial design problem. For higher values above 1.0, the first model has a higher impact on the MMO result.

3.2 Nonlinear Analysis and Nonlinear Optimization of the Safety Measure

The objective of the nonlinear optimization is to find the best value of $a$, for which the design has optimized energy absorption and fail-safe behaviour in the worst case loading scenario. As the plastic deformation and failure behaviour is concerned, a nonlinear and explicit analysis is performed in the design model. The topology result of the MMO is a continuous density field, hence a filtering and smoothing operation is needed to obtain a discrete geometry.

$$M_d = \{ M \in \Omega | \rho > \rho_c \}$$

(4)

In Eq. (5) $M_d$ is the set of elements considered for the geometry reconstruction for the nonlinear analysis model. All elements with an element density $\rho$ greater than a defined limit density $\rho_c$, which should be in accordance with the volume constraint, are considered.

A safety measure for energy absorption and fail-safe behaviour in the worst case scenario is defined as follows:

$$S(a) = E_{d,stat}^V(a) + \beta E_{d,fail}^V(a)$$

(5)

The safety measure $S(a)$ is defined as a relation of the deformation energy in the nonlinear analysis. The latter being a function of $a$, as the geometry for the nonlinear analysis depends on $a$. $E_{d,stat}^V$ and $E_{d,fail}^V$ are the volume specific total deformation energy and the volume specific deformation energy after first rupture of one structural member until total failure. The scalar parameter $\beta$ is a weighting factor for the fail-safe behaviour of the structure.

The loading scenario in the nonlinear analysis is adjusted in comparison to the static linear SIMP topology optimization and MMO, as the structure is to be analyzed in the nonlinear and failure regime. For this purpose, in the example tension beam problem a constant deformation velocity in loading direction is applied at the force loaded nodes to represent a worst case overloading scenario. This matches the loading condition in the tension testing, described in the next section. Still, this adjustment of boundary conditions in the nonlinear model and its correlation requirements in relation to the initial design problem needs to be further addressed for other design problems.

The nonlinear optimization is performed with a golden-section search. The optimized value of $a$ that has been found after 7 iterations in the nonlinear optimization for the described design problem is $a_{opt} = 17.23$.

4. Experimental Validation by Testing of 3d Printed Structures

For validation, two design solutions for the tension beam are designed for 3d printing with equal mass and tested in a tensile test experiment. The aluminum alloy Scalmallox is used with the mechanical values described in the last section. Still, it is to be considered that the mechanical properties for the 3d printed material can be inhomogeneous and depend on the cross sectional dimension. Also, material defects were not considered. The specimen was fixed on the left hand side and loaded with a uniform traverse speed of 1 mm/s. The force that has to be applied for impinging the uniform speed on the specimen is measured in the experiment. In Figure 4. and Figure 5. the tensile test results are illustrated. Marked in white are the rupture areas and sequence of rupture by numeration. The standard SIMP
design shows a total collapse directly after the first rupture occurs, due to the simultaneous breakage of the two marked beams. Instead, the design, which was optimized by the proposed design model, shows the intended fail-safe behaviour. First, the beams with smaller cross section get ruptured and later on the larger beam and the whole structure collapses at a higher deformation level.

![Figure 4](image_url)

Figure 4. 3d printed specimen after tensile testing. Left: Specimen design derived from the standard SIMP topology result. Right: Specimen designed and optimized with the proposed design model.

In Figure 5, the results of the measured and simulated applied force for the impinging deformation in simulation and experiment of the tensile test are illustrated. Table 1. shows a summary of the measured and simulated data. The stiffness $K$, being the linear correlation of force $F$ and deformation $u$ in the elastic regime, differs strongly between experiment and simulation. This might result from the deviation of the modeled elastic properties of the material and neglected strain rate effects. Also the simulation and experimental data for the standard SIMP topology in a) differs in the plastic regime. A reason could be the complexity of the topology and notch effects, as well as modelling discrepancies in the plastic regime of the material. The overall behaviour of the simulation was successfully validated with the experiments. The energy absorption was significantly improved as well as the max. load capacity before breakage. Also, fail-safe behaviour was successfully implemented with the design model. Interestingly, also the stiffness in the elastic regime is slightly higher than with the standard SIMP topology optimization. This might be due to the difference in stiffness evaluation in the tensile test compared to the compliance calculation in TO.

![Figure 5](image_url)

Figure 5. Nonlinear simulation and experimental testing of 3d printed specimen. Left: Design from standard SIMP topology optimization. Right: Design result with $a_{eq}=17.23$ from the proposed design model.

<table>
<thead>
<tr>
<th>Design</th>
<th>$K$ [kN/mm]</th>
<th>$F_{max}$ [kN]</th>
<th>$E_{tot}$ [J]</th>
<th>$E_{eq}$ [J]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Std. SIMP sim.</td>
<td>111.6</td>
<td>42.6</td>
<td>122.9</td>
<td>0</td>
</tr>
<tr>
<td>Design model sim.</td>
<td>112.4</td>
<td>46.8</td>
<td>307.3</td>
<td>40.3</td>
</tr>
<tr>
<td>Std. SIMP exp.</td>
<td>60.54</td>
<td>42.97</td>
<td>191.87</td>
<td>0</td>
</tr>
<tr>
<td>Design model exp.</td>
<td>65.25</td>
<td>47.87</td>
<td>312.17</td>
<td>52.33</td>
</tr>
</tbody>
</table>
5. Conclusions

A design model was proposed to improve the energy absorption and fail-safe behaviour of topology optimized structures with the potential of reducing safety factors for worst case scenarios. The implementation was successfully demonstrated on a tensile beam design problem and solutions were 3d printed and tested in a tensile test experiment. The energy absorption was improved by a high margin and also the optimized structure was able to withstand significant loads after the first rupture and thus failed in a safe manner. The findings demonstrate the high potential of improving the topology of mechanical designs with a focus on safety and robustness apart from solely compliance based objectives. Also, there must not necessarily be a tradeoff in stiffness and mechanical effectiveness for static load cases, when optimizing for safety and robustness.

The design model is implemented in an automated way on basis of commercial software tools. Still, there are manual and empirical design steps that need further improvement. Especially the derivation of models for the MMO is very delicate to the success of the optimization approach. Also, the correlation of boundary conditions in the initial model and the nonlinear analysis needs further investigation. As for now, the design model is conceived for 2d design problems with unidirectional loads. Extension to 3d and more complex boundary conditions in commercial applications will be the topic of future research.

References

VARIABLE SCREENING OF MULTIPLE RESPONSE SYSTEM USING NONPARAMETRIC NEIGHBORHOOD COMPONENT FEATURE SELECTION

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Abstract

As the number of design variables increases, the convergence rate of optimization tends to decrease, and the number of function calls and design change cost will increase. To resolve these problems, variable screening techniques are being used to analyze significant design variables, and to reduce the number of design variables used for optimization and design. This study proposes nonparametric Neighborhood Component Feature Selection (nNCFS) by removing parameter determination process of NCFS, and by modifying optimization problem of NCFS. Additionally, new weight allocation methods named as violation-based method and objective-oriented method are proposed to make decision which design variables need to be selected considering multiple responses and design optimization problems. The proposed methods are applied to mathematical examples and an engineering problem which is an autonomous platform demonstrator called as unmanned ground vehicle. Results are compared based on evaluation criteria which are robustness, computational cost, and feasibility/improvement of optimum designs. In many cases, nNCFS shows better results than NCFS. Also, feasible and improved optimum designs are obtained through violation-based method and objective-oriented method using fewer number of significant design variables respectively.

Key words: Multiple Response System, Variable Screening, Feature Selection, Nonparametric Neighborhood Component Feature Selection, Screening Measure

1. Introduction

Analysis of variance which is often used for variable screening is a hypothesis test that statistically analyzes a influence of response to changes in design variables using level-specific input data such as orthogonal array. However, if there is no level-specific input data, additional time can arise to obtain the input data. Also, surrogate-model-based design optimization is frequently performed because it is difficult to perform optimization using computationally expensive models. To build the surrogate model, input data should be generated via design of experiments considering the property of space-filling such as optimal Latin hypercube design and maximin distance design. Although level-specific input data has the advantage of analyzing the significant design variables using a small number of data, there is a limit which could be not suitable for surrogate modeling.

This study adopts Neighborhood Component Feature Selection (NCFS) in feature selection technique among machine learning to perform variable screening regardless of input data’s condition. NCFS solves a multi-objective optimization problem, which derives optimum weights that minimizes mean loss of neighborhood component analysis (NCA) regression model and prevents overfitting by the regularization term. The objective function consists a parameter (λ) which is a ratio of the NCA regression model’s mean loss and the regularization term. The parameter is determined by computationally expensive process such as k-fold cross validation and grid search prior to performing NCFS. Therefore, this study proposes nonparametric NCFS (nNCFS) by removing the parameter determination process and modifying optimization problem of NCFS. Additionally, screening measure is needed to perform decision making which design variables need to be selected considering multiple responses and design optimization problem. Accordingly, violation-based method and objective-oriented method are proposed to allocate weights for each response. The proposed methods are compared through mathematical examples and an engineering problem based on robustness, computational cost, and feasibility/improvement of optimum designs.
2. Variable Screening Using Feature Selection

2.1 Feature Selection
Feature selection is generally classified into filter, wrapper, and embedded methods [1]. Filter method statistically analyze significant design variables using input and output data. Filter method includes analysis of variance, Chi-square test, Pearson’s correlation coefficient, and linear discriminant analysis. Wrapper and embedded methods analyze significant design variables by generating design variable subsets, constructing the machine learning model, and evaluating the design variable subsets using the machine learning model. Wrapper and embedded methods are slightly different in the analysis process, but they are similar methods in the view of using machine learning models.

2.2 Neighborhood Component Feature Selection (NCFS)
NCFS [2] was developed based on neighborhood component analysis (NCA). NCA is simpler and more efficient than other algorithms such as support vector machine, artificial neural network, and nearest neighbor algorithms. NCFS constructs NCA regression model and performs variable screening by deriving optimum weights that minimizes mean loss of the NCA regression model using leave-one-out cross validation. Optimization problem of NCFS is formulated as follows:

$$\min_{w} \frac{1}{n} \sum_{i=1}^{n} L_i + \lambda \sum_{i=1}^{n} w_i$$

(1)

where \( n \) is the number of data, \( n_v \) is the number of design variables, \( w_i \) is a weight of NCA regression model at \( r \)-th design variable, \( L_i \) is a loss of NCA regression model at \( i \)-th data, and \( \lambda \) is a parameter. The second term of Eq. (1) which is called regularization term prevents overfitting of the NCA regression model. However, determination process of \( \lambda \) needs lots of computational cost such as \( k \)-fold cross validation and grid search prior to performing NCFS. Also, as influence of boundary of \( \lambda \) has been studied empirically [2], the variable screening results can be varied depending on the boundary.

2.3 Nonparametric Neighborhood Component Feature Selection (nNCFS)
Optimization problem of nNCFS is proposed as follows:

$$\min_{w} \frac{1}{n} \sum_{i=1}^{n} \frac{L_i^*}{L_i^*} + \sqrt{\sum_{i=1}^{n} w_i^2}$$

(2)

where \( L_i^* \) is a normalization value formulated as Eq. (3),

$$L_i^* = \frac{L_i^{\text{init}}}{n_d}$$

(3)

where \( L_i^{\text{init}} \) is the initial loss of the NCA regression model at \( i \)-th data, nNCFS removes the determination process of \( \lambda \) which adjusts scales between the mean loss term and the regularization term. Then, the mean loss term is normalized via the initial data. As mentioned above, the regularization term prevents overfitting, but it often makes a few design variables having a very small weight, which is hard to interpret the results. Therefore, to make weights of insignificant design variables exactly zero, square root is applied to the regularization term.

3. Screening Measures for Significant Design Variable of a Multiple Response System

3.1 Multi-criteria Decision Making (MCDM)
In renewable and sustainable energy areas, certain energy planning enormously influences on societies such as social, economic, and environment. For these reasons, decision making for renewable and sustainable energy planning considering multi-criteria has been studied, which is known as multi-criteria decision making (MCDM) [3]. MCDM is performed based on the grouped decision matrix as follows:
Table 1. Grouped Decision Matrix

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Weight</th>
<th>Alternative</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>...</th>
<th>$A_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$w_1$</td>
<td>$s_{11}$</td>
<td>$s_{12}$</td>
<td>...</td>
<td>$s_{1n}$</td>
<td></td>
</tr>
<tr>
<td>$C_2$</td>
<td>$w_2$</td>
<td>$s_{21}$</td>
<td>$s_{22}$</td>
<td>...</td>
<td>$s_{2n}$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>$C_i$</td>
<td>$w_i$</td>
<td>$s_{i1}$</td>
<td>$s_{i2}$</td>
<td>...</td>
<td>$s_{in}$</td>
<td></td>
</tr>
</tbody>
</table>

where $n$ is the number of alternatives, $n$ is the number of criteria, $s_{ij}$ is score of $i$-th criterion at $j$-th alternative, $w_i$ is the weight of $i$-th criterion. MCDM process consists of multi-criteria decision analysis (MCDA) which is how to calculates weights of criteria and scores considering multi-criteria and allocating weights which is how to allocate the weights of criteria.

3. 2 Weight Allocation Methods for a Design Optimization Problem

The process of selecting significant design variables considering multiple responses is similar to MCDM. There are several weight allocation methods for sustainable energy systems, but adequate weight allocation methods are needed for design optimization problem. Violation-based method and objective-oriented method are proposed. Briefly, violation-based method allocates more weights on the violated responses to obtain a feasible design and objective-oriented method allocates more weights on the objective responses rather than constraint responses to obtain an improved design.

4. Applications

Proposed nNCFS and weight allocation methods are applied to mathematical examples [4-5] and an engineering problem [6]. Descriptions of the problems are summarized in Table 2.

Table 2. Descriptions of the Mathematical Examples and the Engineering Problem

<table>
<thead>
<tr>
<th>Problem</th>
<th>Mathematical</th>
<th>Engineering</th>
</tr>
</thead>
<tbody>
<tr>
<td># of DVs</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td># of responses</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>Initial design</td>
<td>Infeasible</td>
<td>Feasible</td>
</tr>
</tbody>
</table>

Results of NCFS and nNCFS are compared at variable screening process. Evaluation criteria for the comparison are robustness and computational cost. Robustness is evaluated by standard deviation of duplication and repetition results, and computational cost is evaluated by mean and standard deviation of the computational time [sec]. At screening measure process, equal weights method and violation-based method, equal weights method and objective-oriented method are compared. Evaluation criterion for the comparison is how well the rank reflected the true significance of objective and constraint responses. Finally, at optimization process, optimization results are compared using significant design variables based on the screening measure results. Evaluation criteria for the comparison are robustness and feasibility/improvement of optimum designs.

5. Conclusions

The nNCFS have similar variable screening results with NCFS. However, robustness and computational cost of nNCFS for variable screening are highly improved in many cases owing to removing the parameter determination process of NCFS. Depending on proposed weight allocation methods, screening measure values have the different trend, and this reflects optimization results. More number of feasible designs and more improved designs are obtained using violation-based and objective-oriented method respectively. The optimum designs are obtained by using a fewer number of significant design variables than equal weights method.
Acknowledgments

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2019R1A2C1007644).

References

FREE MATERIAL OPTIMIZATION OF MULTILAYER
COMPOSITE MATERIALS

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Abstract

The use of composite materials in various types of devices have increased steadily in the past decades. Hence it is worth investigating into the structural behavior of these materials. The optimization of stacking sequences in multilayer composite materials opens a large field of improvements in industrial applications of composite materials. The center of the investigation is set on fiber reinforced laminate plies, which are examined in the case of the linear anisotropic theory of elasticity. The natural way of modeling typical domains of application is the deployment of geometrical non-linear shell elements.

A very auspicious way for optimization of these composite laminates is the implementation of Free Material Optimization. In this optimization procedure arbitrary values of parameters of the elasticity matrix are obtained under constraints that only claim the elasticity matrix to be symmetric and positive semi-definite. This is controlled by restricting the eigenvalues of the material stiffness matrix. Also the trace of the elasticity matrix is restricted to avoid the total stiffness of the material to get unnaturally large.

The sensitivity information is received by the variational approach for sensitivity analysis which represents the most efficient known way of computing sensitivity information.

The Free Material Optimization for multilayer materials yields an optimal distribution of material parameters in each single layer of the stacked ply that can be taken into account for further optimization procedures to choose the best fitting material for plies of the considered stack.

In the presentation the application of Free Material Optimization with control of eigenvalues in material stiffness matrices will be demonstrated by simplified examples.

Key words: Free Material Optimization, Laminates, Composites

1. Introduction

The optimization of multilayer composite materials is a field in which many research projects achieved great improvements during the last years. The most important application area is thin walled structures like laminates of very high strength fiber materials like they are often used in aerospace or wind energy applications. These thin walled structures are normally directly connected to the mechanical field of shell theories because only shells are able to represent curved structures. It is mostly not reasonable modeling all the single plies of a laminate shell with its own elements so that waves were found to take preintegration schemes other the thinkness of the laminates to model the whole structure. Mostly this is also the starting point for optimization, where the reducing of element number also reduces the number of necessary degrees of freedom in design so that the numerical afford can be limited.

The basic concept of Free Material Optimization (FMO) was introduced in the 1990s c.f. [5]. Since then, there have been many research projects on FMO. The fundamental idea of FMO is the Optimization of a material tensor, which can vary in each single Gaussian Point of the design domain. There is only one major restriction to the optimization so that the material tensor has to be positive semidefinite. The so obtained values of the material tensor are characterizing the distribution of material density and local material properties in an indirect way. Results of FMO normally are irrelevant for manufacturing of structures because it is technically costly to build structures out of often
changing materials. It can be reasonably assumed that the optimized results of FMO show the best possible solution
within the design space so that they can be seen as a benchmark for comparison to simplified optimization approaches
which might be more valuable with respect to manufacturing constraints.

The most promising approach with the idea of FMO lies in patch field analysis which identifies larger areas in the
structure with not significantly changing stress fields in which the only one set of material parameters which means to
assume the same material tensor in each Gaussian point to ensure the manufacturability of structures. The first step
into this approach is dividing the structure of different numbers of arbitrary chosen equally sized patches and thereby
reducing the number of independent design parameters. The so gained results can be compared to the benchmark
solution with different material tensors in each point.

Due to the restrictions on the material tensor to be positive semidefinite the obtained problem is a nonlinear
semidefinite optimization problem [1]. There have been many research projects mostly from the perspective of
mathematics on finding algorithms (called semidefinite programming), which efficiently solve these types of
problems. For engineering applications it might be appropriate to choose simplified optimization algorithms to get
better glimpse into the processes during the optimization. With that requirement the Method of Moving Asymptotes
(MMA) [4] seems to be an appropriate alternative to the nonlinear semidefinite optimization algorithms. Controlling
the eigenvalues of the material tensors so that they are all positive nonzero values fulfills the controlling of positive
semidefinite.

The considered optimization problem is a minimum compliance problem with restriction to the maximal stiffness.

2. Variational Sensitivity Analysis

The design sensitivities, which are used for the optimization, are gained by the variational approach by [2]. Thereby
the total variation of the weak form of static equilibrium is divided into one part with respect to the displacement field
and one part with respect to the change of design parameters.

\[ \delta R = \delta s \cdot R + \delta s \cdot R \]  

There the total derivative of the residuum with respect to the displacement field is known as tangential stiffness the
total derivative with respect to the design change is calculated to throw;

\[ \delta R = \frac{dR}{ds} \delta s = (\frac{\partial R}{\partial s} + \frac{\partial R}{\partial u} \frac{du}{ds}) \delta s = 0 \]  

In this equation the so called pseudo load matrix \( P \), the stiffness matrix \( K \) and the sensitivity matrix \( S \) can be identified.

\[ P = \frac{\partial R}{\partial s}, \quad K = \frac{\partial R}{\partial u}, \quad S = \frac{du}{ds} = -K \cdot P \]  

3. FMO Problem Formulation

The optimization problem is focused on a case of two-dimensional linear elasticity. So that only in plane stresses can
occur. Assuming a two-dimensional material law reduces the number of independent material parameters in each
Gaussian point from a total number of 21 (symmetric tree-dimensional material) to a number of only 6 independent
parameters.

3.1 Material Parameter

In the approach of FMO, the material parameters are normally not restricted by physical properties. It does make
sense to restrict the design domain by boxed constraints which are calibrated by reasonable values which are known
from the typical physical behavior of materials like Young’s module or Poisson’s ratio. But there are no restrictions to
the relation between different parameters which are part of the material tensor so that the can vary completely
independent from each other. That leads to the fact that normally no material exists which can exactly fulfill all the
optimized material parameters. This is not a big problem if the focus is laid on multilayer materials. In this case the
material tensor of a laminate is the result of the preintegration of material properties in each single layer other the
whole thickness.

\[
\tilde{E} = \begin{bmatrix}
E_1 & E_2 & E_3 \\
E_2 & E_1 & E_4 \\
E_3 & E_4 & E_5 \\
\end{bmatrix}, \quad \text{with} \quad \tilde{E} = \tilde{E}^T \geq 0
\]  

(4)

3.2 Finite Element Formulation and Sensitivity Analysis

For the finite element formulation linear elasticity in a two-dimensional case is assumed. The weak form of static equilibrium can be obtained.

\[
R(u,v) = \int_{\Omega} \mathbf{a}(u,s) : \mathbf{\varepsilon}(v) \, d\Omega - \int_{\Gamma} \mathbf{b} \cdot \mathbf{v} \, d\Gamma - \int_{\Gamma} \mathbf{T} \cdot \mathbf{v} \, d\Gamma
\]  

(5)

With this weak form the pseudo load matrix eg. Eq. (3), which is significant for the optimization can be calculated by variation. In this weak form only the term of stresses is dependent on the design variables,

\[
P = \frac{\partial R}{\partial s} = \int_{\Omega} \frac{\partial \mathbf{a}(u,s)}{\partial s} : \mathbf{\varepsilon}(v) \, d\Omega = \int_{\Omega} \frac{\partial \tilde{E}}{\partial s} (u) ; \mathbf{\varepsilon}(v) \, d\Omega
\]  

(6)

3.3 Optimization Problem

The optimization problem is of the type minimization of compliance that means the maximization of global structures stiffness. The compliance C and its gradient with respect to changes in design can be calculated by known quantities.

\[
C(s) = -2\Pi(s) = u^T f \quad \nabla C(s) = \frac{\partial C(s)}{\partial s} = -2 \frac{\partial \Pi(s)}{\partial s} = -u^T P
\]  

(7)

Restrictions are given to the material tensor which must stay positive semidefinite during the optimization process.

\[
E = \tilde{E}^T \geq 0
\]  

(8)

To circumvent the necessity of choosing nonlinear semidefinite optimization algorithms that maintain the positive semidefinite in each step automatically this restriction must be superseded by another restriction which can be implied into the MMA algorithm. It is known that each tensor whose eigenvalues are all larger or equal to zero is positive semidefinite.

\[
\lambda_i \geq 0, \quad i = 1, 2, 3, \quad \text{with} \quad (E - \lambda_j I) v_j = 0, \quad v_j^T v_j = 1
\]  

(9)

Additional to this restriction the change of eigenvalues during the optimization is needed. The gradient of the eigenvalues with respect to a change in design can be calculated by [3]:

\[
\nabla \lambda_i = \frac{d\lambda_i}{ds} = v_j^T \frac{dE}{ds} v_j
\]  

(10)

This leads to the optimization problem:

\[\begin{align*}
\text{minimize} \quad & C(s) \\
\text{subject to} \quad & -\lambda_i \leq 0, \quad i = 1, \ldots, 3 \\
& s_i^L \leq s_i \leq s_i^U, \quad i = 1, \ldots, 6 \\
& \rho^T \leq \sum_{j=1}^{6} \lambda_i \leq \rho^T \\
& s_i^L = 0, \quad 0.1 s_i^U \\
& s_i^U < \rho^T
\end{align*}\]  

(11)

In this problem formulation the boxed constraints for all material parameters are included. Additionally there is a restriction to the minimal and maximal occurrence of material. Therefore a pseudo density value is chosen. This pseudo density is the sum of the eigenvalues multiplied by the thickness of the layers. Furthermore the lower limit of design parameters is restricted to be not smaller than 1% of the upper limit to avoid numerical errors due to scaling effects.

Also the minimal pseudo density should not be smaller than the highest values of possible design variables to avoid the restriction of pseudo density to become invalid.
4. Numerical Example

To show the optimization scheme is working precisely a numerical example is consulted. Therefore a cantilever beam, which is clamped on one edge and loaded on the other, is chosen. For the initial design and the affecting force the following values are resumed:

\[
E = \begin{bmatrix}
50000 & 1000 & 1000 \\
1000 & 50000 & 1000 \\
1000 & 1000 & 1000
\end{bmatrix} \text{kN/cm}^2
\]

\[
\tau = 0, \text{ 1 cm}
\]

\[
F = 10 \text{ kN}
\]

The boxed constraints are defined as:

\[
300 \leq \sum_{i=1}^{3} \lambda_i \leq 12000 \text{ kNcm}^2/\text{cm}^3 \quad 1000 \leq E \leq 100000 \text{ kN/cm}^2
\]  

With these assumptions the optimization provides results, which show the success and the improvement of structural behavior by the procedure. Most significantly this can be seen in the change of displacement between the initial and the optimized design.

5. Patch Analysis

To reduce the total number of design variables it is advised to divide the total structure in patches and assume the material parameters to be the same in all points within this patch. So numerical effort is rising with the number of different patches. But also the quality of results declines with a low number of patches. The aim of patch analysis is to find the best number and the best distribution of patches so that the results of optimization give satisfactory values while the numerical afford is still manageable. The approach of dividing the structure in a fixed number of equal sized patches is only the first step into this field of patch analysis and can only give hints upon the effects by reducing the
number of design variables.

In Table 1 it is shown, that there is a very significant effect by the total number of patches to the optimized compliance, the computing time and the number of iteration, which is needed for convergence. For a low number of patches, however, the number of finite elements is the same for all models, the number of necessary iteration and the computing time does not change too much. Also the compliance is decreasing slowly. For a very high number of patches, especially with one patch for each finite element, the number of iteration and most striking the computing times are rising significantly. Bearing in mind that the number of design variables, which are included in the sensitivity analysis and optimization is 768, explains the high numerical effort.

<table>
<thead>
<tr>
<th>Number of patches</th>
<th>Compliance</th>
<th>Number of iteration</th>
<th>Computing time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9687</td>
<td>153</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>0.9653</td>
<td>157</td>
<td>1.28</td>
</tr>
<tr>
<td>4</td>
<td>0.9563</td>
<td>178</td>
<td>1.43</td>
</tr>
<tr>
<td>8</td>
<td>0.8854</td>
<td>217</td>
<td>1.79</td>
</tr>
<tr>
<td>32</td>
<td>0.8471</td>
<td>414</td>
<td>3.45</td>
</tr>
<tr>
<td>128</td>
<td>0.8392</td>
<td>630</td>
<td>158,77</td>
</tr>
</tbody>
</table>

6. Conclusions

It could be shown, that the FMO approach with the MMA algorithm and controlling the eigenvalues of material tensors provides a significant improvement in the structures stiffness, although not the commonly used semidefinite optimization algorithms were implied. The analysis of impact by dividing the structure in a smaller number of patch fields showed, that it is worth investigating in patch analysis to identify the best suitable patch sizes and distributions for the wanted results. At this point there is much more research to do.

Acknowledgments

The founding of this project by the German Research Foundation and the provision of the MMA algorithm by Krister Svanberg [4] is kindly acknowledged.
References

FAST TOPOLOGY OPTIMIZATION FOR RESONATING STRUCTURES VIA GENERALIZED INCREMENTAL FREQUENCY METHOD AND MODAL SUPERPOSITION-BASED MODEL ORDER REDUCTION

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Abstract

This contribution presents an approach to decrease the computational effort of the generalized incremental frequency method, initially used to obtain designs with minimal dynamic compliance under external harmonic excitation. This goal is achieved by utilizing a modal superposition-based model order reduction technique. We show that the inclusion of model order reduction accelerates the Generalized Incremental Frequency (GIF) method without considerable drawbacks. Furthermore, it is a first attempt to use the GIF method for targeting specific resonance frequency instead of only minimizing dynamic compliance at a prescribed frequency value.

Key words: Topology Optimization, Generalized Incremental Frequency Method, Model Order Reduction, Modal Superposition

1. Introduction

Recently, energy harvesting has received a considerable amount of attention, due to the increasing need for battery free but independent energy supply for perpetually powered systems. Energy harvester devices transform environmental energy into electrical power. The design of these multi-physical devices traditionally involves parameter optimization, which is highly reliant on the designers’ intuition, making it hard to come up with revolutionary designs. This is especially true for multi-resonant designs. Topology Optimization (TO) bears the potential to tackle this issue. However, TO algorithms are known to have very high computational costs as a Finite Element (FE) analysis of the structure is usually needed in every iteration. This computational effort can be significantly reduced by the mathematical methodology of Model Order Reduction (MOR). There have been several publications addressing combined approaches of TO and MOR for minimizing static and harmonic compliance [1-5]. Most of them generate Reduced Order Models (ROMs) in every TO iteration and therefore we refer to them as on-the-fly-approaches. A review on the influence of different MOR techniques on TO can be found in [6].

The Generalized Incremental Frequency (GIF) method was introduced in [7]. The method is based on changing material distribution to minimize the dynamic compliance of undamped elastic continuum structures for given external harmonic excitations. Commonly, the design goal in such cases is to drive the structures resonance frequencies as far away from the excitation frequency as possible. Our goal, however, is to have the resonant frequencies placed at specific locations of interest, e.g. the structures resonance frequency should match the excitation frequency. This frequency matching is the typical design goal for vibration-based energy harvesting [8-10] using the piezoelectric effect. Apart from matching resonance frequencies, the corresponding mode shapes are of interest as well. There is no benefit from the mere existence of an eigenmode, which makes pure eigenfrequency optimization unfeasible. The eigenmode needs to be excitable (and thereby practically usable) in the intended application scenario and the mode shape shall exhibit desirable deformation within the spatial region of interest with defined oscillation amplitudes. Ideally, these characteristics have to be preserved under varying environmental conditions. The first goal can be achieved by utilizing the property of the GIF method to steer the structures resonance frequencies up to or down by minimizing its dynamic compliance for a specific array of external driving frequencies, which will be the focus of this contribution. The remaining goals will be added to the algorithm in future works. Finally, we combine the adapted
GIF method with modal superposition-based MOR (cf. e.g. [11]), similar to e.g. [12] and [13], where the authors have combined modal superposition-based MOR with classical TO. As the GIF method requires an eigenmode analysis in every optimization loop, the addition of modal superposition-based MOR does not require extra computational effort.

Section 2 introduces basic ideas of the GIF method, MOR methodology and our combined approach. In section 3, some numerical examples on the standard test beam structure are shown. We conclude and give a brief outlook in the final section 4.

2. Fast Topology Optimization for Resonating Structures

2.1 General Incremental Frequency (GIF) Method

An optimization problem of minimizing dynamical compliance of an undamped continuum structure can be written as:

$$\min_{\rho} \{ F = C_{s}\}$$

subject to:

$$C_{s} = |p^T x|,$$

$$(K - \omega^2 M) x = p,$$

$$\sum_{i=1}^{N} a_{V_{i}} - V^* \leq 0, (V^* = a V_{s}),$$

$$0 < \rho_{\text{min}} \leq \rho \leq 1,$$

where $C_{s} \in \mathbb{R}$ is the dynamic compliance, $p \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$ are the amplitude and the prescribed frequency of external excitation, and $x$ describes the nodal displacement, $K \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{n \times n}$ respectively the structural stiffness and mass matrix resulting from e.g. finite elements method, $V_{i} \in \mathbb{R}$ denotes the volume of each element and $a \in [0, 1] \subset \mathbb{R}$ the predefined volume constraint ($V^* = a V_{s}$), which is a fraction of the full volume $V_{s} \in \mathbb{R}$. The optimization parameter $\rho \in \mathbb{R}$ describes the density of each element and ranges from a minimal value $\rho_{\text{min}}$ (slightly larger than 0, because of numerics) to 1. $C_{s}$ is chosen as a norm since the dynamic compliance can become negative if the excitation frequency $\omega$ is larger than the first resonance frequency. Furthermore, the goal function is chosen as $C_{s}$ for better differentiability.

The GIF method solves the optimization problem (1) using the classical SIMP (Solid Isotropic with Penalization) approach, i.e. $K_{s}(\rho_{s}) = \rho_{s} K_{s}$ and $M_{s}(\rho_{s}) = \rho_{s} M_{s}$ with $K_{s}$ and $M_{s}$ being the respective matrices for a full element and $q$ being the penalization power (usually set to 3). The method uses a series of excitation frequencies $\omega_{s}$, which are chosen to be slightly different from the resonant frequency of the structure, for which one wishes to optimize the dynamical compliance. Thereby, the resonance frequency of the structure is being alternated. $\omega_{s}$ is increased or decreased incrementally in each optimization loop, from the initial value towards the prescribed final value $\omega_{s}$. Furthermore, the same procedure is performed with different initial excitation frequencies, to obtain the global optimum. We make use of this property and change the choice of $\omega_{s}$ in such a way, that the optimization now steers the structure’s resonance frequency towards the prescribed value $\omega_{s}$. For example, let $\Omega_{k}$ be the $k$-th resonance frequency of the structure under optimization. If $\Omega_{k} < \omega_{s}$, then set $\omega < \Omega_{k}$ and vice versa under the condition $\Omega_{k-1} < \omega_{s} < \Omega_{k+1}$. Finally, we assume that increasing the distance between $\omega_{s}$ and $\Omega_{k}$ will decrease the change of $\Omega_{k}$ within an optimization step. Therefore, we introduce a parameter inversely proportional to this distance to control the travel distance of $\Omega_{k}$.

The optimization itself can be performed with most Non-Linear Programming (NLP) algorithms. The two most common NLP used for TO is Optimality Condition (OC) [14] method and the Method of Moving Asymptotes (MMA) [15]. As NLP is not the focus of this contribution, we refer to the aforementioned publications for further details.

2.2 Model Order Reduction (MOR)

Let $M, K$ and $p$ be given as in equation (1). The nodal displacement $x$ of the structure complies with the following
dynamical system \( \Sigma \):

\[
Kx + Mx = pu,
\]
\[
y = Cx,
\]
where \( y \in \mathbb{R}^m \) denotes the user defined output vector, which is gathered by the output matrix \( C \in \mathbb{R}^{m\times n} \). \( u \) is the system input. For a harmonic external excitation with frequency \( \omega \), \( u = e^{i\omega t} \). The methodology of MOR aims to approximate the input and output behavior of the system with a ROM \( \Sigma \) of the form:

\[
K_x + M_x = p_u
\]
\[
y = Cx.
\]
The ROM is generated by projecting the full system onto a significantly lower dimensional subspace with an appropriate projection matrix \( V \in \mathbb{R}^{n\times m} \), such that \( (K,M) = V^T(K,M)V \in \mathbb{R}^{m\times m} \), \( x = V^Ty \in \mathbb{R}^m \) and \( C = CV \in \mathbb{R}^{m\times p} \). If the projection matrix \( V \) is chosen to be a \( q \)-dimensional eigenbasis of \( \Sigma \), the MOR method is referred to as modal superposition. As the GIF method requires an eigenfrequency analysis in each optimization loop, we use the eigenvectors obtained from this analysis for modal superposition-based MOR. We solve the significantly lower dimensional equilibrium equation:

\[
(K - \omega^2 M)x = V^Tp,
\]
instead of the full order equation (1). The approximation error when solving the projected equilibrium equation is negligible as long as the columns of \( V \) correspond to the dominant eigenmodes of the structure, which can be identified by different measures, e.g. \[16\]. Note that the projection of the system matrices does not need to be computed explicitly, as \( K_s \) is obtained by the eigenanalysis, similar to the algorithm introduced in \[13\]. The sensitivity can be computed as usual, either by also projecting the sensitivity \( K_s \) onto the reduced order subspace or via the up-projection of the reduced order displacement vector \( x \), (i.e. \( x \approx Vx \)). As most structures have only a small number of dominant eigenmodes, \( r \ll n \) applies. Therefore, by solving the lower dimensional projected equilibrium equation instead of the original one, the computational effort can be significantly reduced in each optimization loop depending on the size of the full model.

3. Numerical Experiments

For the numerical study of our algorithm, we implement it in MATLAB and consider a simple double clamped beam as depicted in Figure 1. The length of the beam is set to 10 and the height is set to 1. The Young’s modulus of the material is set to 10^10 and the density is set to 1. For optimization, the beam is divided into 1000 \( \times \) 1000 elements. We choose \( \omega_0 = 100 \), \( \|p\| = 1000 \), set the desired volume fraction to 0.5 and “push” the first resonant frequency of the structure \( \Omega \), which values at 96.4 for the initial design, beyond this prescribed value \( \omega_0 \). We choose to evaluate the first 32 eigenvalues similar to the suggested number in the original publication of the GIF method \[7\]. We only consider the first frequency bracket as the same results are expected to apply for all brackets \([\Omega_{r-1}, \Omega_r]\).

![Figure 1. Admissible design domain of the double clamped beam. The red arrow depicts the force of the external excitation.](image1)

![Figure 2. The OC optimization results after 100 iterations. The dynamic compliance for \( \|p\| = 1000 \) values at \( C_{d,\text{final}} = 148.8 \).](image2)

\[1\] All properties are given in SI units.
Figure 3. The OC optimization results after 100 iterations with MOR using ROMs with respectively 8, 16 and 32 DoF (top to bottom) and the same settings as in Figure 2. The dynamic compliances of the respective final structures are:
\[ C_{d,8} = 249.8, \quad C_{d,16} = 217.2 \quad \text{and} \quad C_{d,32} = 160.2. \]

Figure 4. The MMA optimization results after 100 iterations without (top) and with (bottom) MOR. The ROM used for optimization has 32 DoF. The dynamic compliances for these structures are respectively \( C_{d,0} = 146.1 \) and \( C_{d,32} = 190.0 \). The optimization using ROMs of 8 and 16 DoF both diverged (\( \rho = 1 \forall \epsilon \)).

In a first step, we evaluate the reduction of computation time due to the introduction of MOR. We use the classical OC optimization. The optimization results of the standard GIF method after 100 iterations are shown in Figure 2. The dynamic compliance of the final structure at \( \omega_0 = 100 \) values at \( C_{d,0} = 143.4 \). Figure 3 depicts the optimization results, when MOR is involved. Using only 8 DoFs, the optimization result is missing some details, especially in the middle of the structure. We expected this kind of deviation due to the single node excitation, and since MOR does not conserve geometrical properties. Its dynamic compliance for an external excitation with \( \| p \| = 1000 \) values at \( C_{d,8} = 249.8 \), which is roughly a 70\% deviation from the non-MOR optimization. Furthermore, the final structures also still have some “grey areas” corresponding to values of \( \rho \) far away from 0 or 1. By raising the size of the ROM to 16, the deviation of the final structure, when compared to the non-MOR result, is significantly reduced. This is confirmed by the value of dynamic compliance; \( C_{d,16} = 217.2 \). Note that, when the dimension of the ROM is further increased to 32, the result is closer to the MMA optimization results depict in Figure 4. The dynamic compliance values at \( C_{d,32} = 160.2 \) for this structure. However, one can observe some artifacts in the middle of the structure, which needs further assessment in upcoming works.

Figure 4 depicts the results of the same optimization when MMA is used instead of OC. The optimization diverged when the involving ROMs with less than 32 DoF, i.e., it has resulted in a full block with \( V_i \). The optimization result of the full order TO and MOR enhanced TO can be considered identical, except, again, for the fine structure in the very middle of the geometry. This, however, leads to a deviation of the dynamic compliance of 30\% with \( C_{d,0} = 146.1 \) and \( C_{d,32} = 190.0 \). An overview of all results can be found in Table 1, which also shows the computation time for all the optimizations. On average, the computation time is reduced by roughly 30\% due to MOR. We expect a further increase for models with significantly higher DoF, e.g., for realistic 3D models. Furthermore, we found that

\( \rho \) \( \forall \epsilon \). The optimizations commonly converge in significantly less than 100 iterations. The large number of iterations was only performed to suppress random CPU events during the optimization.
80% of the computation time is spent on the eigen analysis. Replacing `eigs()` with a more efficient algorithm or using a more efficient language than MATLAB will significantly increase the benefit of MOR.

<table>
<thead>
<tr>
<th>CPU time [s]</th>
<th>NLP</th>
<th>DoF</th>
<th>Optimized $C_{\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1165.3</td>
<td>OC</td>
<td>201798</td>
<td>143.4</td>
</tr>
<tr>
<td>1102.8</td>
<td>OC</td>
<td>32</td>
<td>160.2</td>
</tr>
<tr>
<td>1100.0</td>
<td>OC</td>
<td>16</td>
<td>217.2</td>
</tr>
<tr>
<td>1091.2</td>
<td>OC</td>
<td>8</td>
<td>249.8</td>
</tr>
<tr>
<td>1303.2</td>
<td>MMA</td>
<td>201798</td>
<td>146.1</td>
</tr>
<tr>
<td>1247.1</td>
<td>MMA</td>
<td>32</td>
<td>190.0</td>
</tr>
<tr>
<td>Did not converge</td>
<td>MMA</td>
<td>16</td>
<td>N. A.</td>
</tr>
<tr>
<td>Did not converge</td>
<td>MMA</td>
<td>8</td>
<td>N. A.</td>
</tr>
</tbody>
</table>

Figure 5. The optimization (MMA) result (bottom) after 300 iterations when using the GIF method to place the first resonance frequency of the system at 150. The change of the structure’s first resonant frequency is plotted against the number of iteration (top).

Figure 6. Same optimization as in Figure 5 after tuning of the penalization power. The final structure after 300 iterations is shown (bottom). The change of the structure’s first resonant frequency is plotted against the number of iteration (top).

In the second step, we use the algorithm to obtain a structure with $\Omega_1 = 150$. Therefore, we now set $\omega_0 = 150$, the frequency of the external excitation $\omega_0$ dependent on the difference between the prescribed target value $\omega_0$ and the first resonance frequency $\Omega_1$ and according to the assumptions we made in section 2. For this experiment, the number of elements remains at $1000 \times 100$ and we perform 300 iterations. As depicted in Figure 5. The final structure contains a large amount of undesired grey areas looking unfinished. Increasing the penalization power ($q = 4, 8$) improves the

© On Intel® CORE™ i5—7600 CPU @ 3.5 GHz, 32 GB RAM, using MATLAB © Release 2017b
result, which is shown in Figure 6. There are still grey areas corresponding to the locations with minimal compliance. During this experiment, the size of the ROM was set to 32.

4. Conclusions and Outlook

In this contribution, we considered a combined approach of TO and MOR; Modal superposition-based MOR has been added to the GIF method, which minimizes the dynamic compliance of undamped elastic continuum structures for given external harmonic excitations. This approach is practical, as an eigenmode analysis is anyhow required by the GIF method in every iteration. In the next step, we adapted the GIF method for resonating structures, to steer the structure’s resonance frequency to a prescribed value. The algorithm still needs improvement since it requires 300 iterations to converge to a usable result (with or without MOR).

For future contributions, we will consider the further adaptation of the algorithm such as an alternative method for eigenanalysis or more efficient programming language. We will also consider alternative optimizers like interior-point etc. and further tune the optimization parameters to achieve better results. As mentioned in the introduction, our ultimate goal is to develop an algorithm suitable for multi-resonant energy harvesting applications. Therefore, we will tune the optimization problem (1) to maximize stress instead of compliance over a frequency band and consider a bimaterial structure. Then, we will finally test the algorithm on real world problems such as multi-resonant energy harvesters from [9], for which MOR has already been successfully tested.

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MODEL PARAMETER IDENTIFICATION BASED ON DIC
DISPLACEMENT DATA COMPRESSION

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Abstract

Nowadays, the development of Digital Image Correlation (DIC) technology has made it possible to measure the full-field of structural deformation, which not only facilitates the direct observation of structural mechanical response. It also provides data support for further study of the physical properties of structures and materials using deformation measurement information. DIC is a full-field measurement system based on the camera and can acquire a huge amount of data. Therefore, extracting useful information and removing redundant noise from huge amounts of data has become an important research topic in recent years. To process large amounts of data, data compression is an efficient means of analysis, which can save a lot of time and cost for calculation. In this paper, Principal Component Analysis (PCA) is used to compress the DIC displacement data, and a model parameter identification algorithm is presented to solve the problem of a large amount of DIC data and difficulty in selecting points. Firstly, in order to reduce the application cost of the huge data measured by digital image correlation technology, the principal component analysis method is proposed to compress the surface deformation data of the structure, so as to reduce the amount of data significantly under the premise of retaining the main features of the surface deformation information. Secondly, the inverse calculation of mechanical model parameters using compressed data is studied. Under the framework of the least square method, it is proved that the target residuals of the inverse model based on compressed data approximate the original data to inverse residuals. Finally, two examples are given to demonstrate the effect of data compression on the inverse calculation of model parameters. The results show that the proposed method can effectively improve the convergence speed of the inverse calculation of mechanical model parameters under the premise of significantly reducing the amount of data used, especially for the inverse calculation of multiple material parameters, with higher accuracy and better stability.

Key words: Digital Image Correlation, Data Compression, Least Squares Method, Model Parameters Identification

1. Introduction

The model parameter identification technology is an effective means to obtain model parameters, which has attracted the attention of researchers in recent years. With the maturity of DIC technology, DIC measurement data and model parameter identification technology are combined into specific engineering problems. This provides a more realistic and powerful means to analyze the mechanical behavior of materials under various complex external loads, and to determine the basic mechanical parameters more comprehensively and accurately.

The model parameter identification is a method of identifying unknown parameters in mechanical models by using experimental data. Wang Bo tested the imperfections of stiffened shells under axial compression. Han et al. studied the parameter inversion of automotive composite materials. For complex mechanical models involving multiple physical parameters, such as composite material mechanics model, elastic-plastic mechanics model and so on, because of the huge amount of data contained in the digital image, if all the measured data are used in the model parameter identification, it will lead to long calculation time and slow convergence speed. Therefore, although DIC technology solves the problem of obtaining structural deformation data, how to use DIC deformation data more effectively has become a new problem. At present, when using DIC data for mechanical model parameter identification, the method of uniform sampling is used to extract discrete data points from the measurement data. However, the spacing between these data points is still uncertain, which can not ensure a more comprehensive access to the main information of the measurement field. If the interval is too large, the amount of data extracted is small, and local information is easily
lost. On the contrary, if the interval is too small, the amount of data extracted will increase, and it will be difficult for subsequent reverse calculation. This is more prominent in the problems with singular deformation, such as the slip of shear bands.

The main goal of the paper is to solve the problem that the DIC measurement data is huge and difficult to select, so as to ensure that the important features of DIC displacement measurement are not lost due to compression. Especially, with the increase of DIC data, the model parameter identification algorithm can reflect the ability of fast convergence, high stability and strong noise processing. In this paper, a data dimension reduction method based on principal component analysis is proposed to compress DIC data, and a model parameter identification algorithm is presented. Firstly, the tensile and bending tests were carried out. Then, the data points of the full-field in the measurement plane were collected, and the data were compressed by selecting the appropriate principal component information satisfying the cumulative variance interpretation rate. Finally, in the framework of the least square method, the minimum difference between numerical results and measured data was used as the optimization goal, and the parameters to be modified are determined by the Gauss-Newton optimization algorithm.

2. Theoretical Research

2.1 Basic Principles of DIC

Among varieties of non-contact full-field optical measurement methods, DIC has been widely used for its advantages of the simple experimental measurement process, strong environmental adaptability, high measurement accuracy and wide application range. For high-quality speckle patterns, the displacement measurement accuracy of DIC can reach 0,01 pixels, and the image data volume is above 100,000.

DIC measurement is to match the geometric points on the digital speckle image of the object surface in different states. The motion of the tracking point can get the deformation information of the object surface. The speckles on the speckle field are randomly distributed, so each speckle is different from the surrounding points. The small area around each point is called a sub-area. The sub-area centered on a point in the speckle field can be used as a carrier for the displacement of the point. By searching and analyzing the movement and change of the sub-area, the displacement of the point can be obtained.

2.2 Compression Process of PCA

PCA is to identify the main features of data. Firstly, the high-dimensional data is projected into the low-dimensional space by a linear transformation, and then the eigenvalue analysis is used to determine the number of principal components that need to be retained, and finally the data is compressed. As a data set preprocessing technology, it can remove redundant information and noise, make data more simple and efficient, and improve computing efficiency. The process of PCA data compression includes three main steps. Take the two-dimensional matrix $U_{mn}$ constructed by DIC measuring displacement field as an example.

$$U_{mn} = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix} \in \mathbb{R}^{m \times n}$$

(1)

Step 1 Data Centralization Processing

$$U_{mn} = U_{mn} - \text{repmat}(\text{mean}(U_{mn}) \cdot m \cdot 1)$$

(2)

In the formula, mean is calculated by column and repmat is copied by matrix block.

Step 2 Eigenvalue Decomposition of Covariance Matrix

$$\sum_{m=1}^{m} \phi = \lambda \phi, \sum_{m=1}^{m} U_{mn} U_{mn}^T$$

(3)

In the formula, $\phi$ is a feature vector and $\lambda$ is an eigenvalue.

Step 3 Data Compression

$$u_j \approx \sum_{j=1}^{m} Z_j \phi_j$$

$$Z_j = u_j \phi_j \in \mathbb{R}^{m \times n}, \phi \in \mathbb{R}^m$$

(4)

(5)
In the formula, the front \( r < n \) order feature vector is intercepted and the original data is compressed.

\[
K_r = \frac{\sum \lambda_i}{\sum \lambda} \geq 80\% \tag{6}
\]

The value of \( r \) should make the explanatory rate of cumulative variance of more than 80\%, so as to ensure that the main information features of the original data are included.

### 2.3 Model Parameter Identification Process

The basic mathematical model for identifying model parameters by the least square method is as follows:

\[
\min \, f_s (A) = \frac{1}{2} \| u_c (A) - u_M \|_2^2 \tag{7}
\]

Where \( A \) is the model parameter to be determined, \( u_c (A) \) is the displacement value calculated by the FEM model, and \( u_M \) is the displacement value measured by DIC.

The PCA processing of data is essentially the projection of the original data in a coordinate system consisting of a set of orthogonal base vectors \( \Phi \), that is, the projection coordinate \( z_i \) of \( u_i \) on the base vector \( \Phi \). The meaning of "compression" is to approximate the original data by intercepting the components with the first few order eigenvalues.

In the framework of the least square method, the following model parameter recognition model is constructed:

\[
\min \, f_s (A) = \frac{1}{2} \| Z_0 (A) - Z_M \|_2^2 \tag{8}
\]

In formula:

\[
Z_0 (A) = u_c (A)\Phi \quad u_c (A) = Z_0 (A)\Phi^T
\]

\[
Z_M = u_M\Phi \quad u_M = Z_M\Phi^T
\]

According to PCA truncation principle, formula (10) and formula (11) are expanded orthogonally:

\[
\| u_c (A) - u_M \|_2^2 = \| z_0 (A)\Phi^T - z_M\Phi^T \|_2^2 + \| Z_0 (A)\Phi^T - r (A)\Phi^T \|_2^2 \approx \| z_0 (A)\Phi^T - z_M\Phi^T \|_2^2 \tag{11}
\]

Then the orthogonality of \( \{ \Phi \} \) is used to obtain:

\[
\| u_c (A) - u_M \|_2^2 \approx \| Z_0 (A) - Z_M \|_2^2 \tag{12}
\]

This ensures the approximation of the new model to the original model. Finally, the Gauss-Newton method is used to solve Eq. (7) and Eq. (8) in the process of model parameter identification. After several iterations, the modified optimization model is obtained and the relevant parameters are concluded.

### 3. Experimental Analysis

#### 3.1 Experimental Setup

The carbon steel \( Q \) was selected from the test material, and the uniaxial tensile test was carried out by a universal testing machine. The displacement information of DIC is measured, the elastic modulus and Poisson ratio of the inverse material are calculated. The DIC instrument includes two four-megapixel cameras, two lighting sources, and Vic-3D measurement systems. The data acquisition process is shown in Figure 1. Set the measurement area, set sub-area 6, step length 7, and select 208 data points to collect displacement information in the sample measurement plane.

![Data acquisition process](image)

Figure 1. Data acquisition process
3.2 Data Compression Based on PCA

The DIC displacement data are compressed by the above theoretical method. For a given measurement area, the measured displacement information can be represented by a two-dimensional matrix. Taking the displacement component of the vertical direction as an example, the surface measurement point data is $26 \times 8$. The data matrix is compressed by principal component analysis, and the first principal component ($26 \times 1$) is selected to meet the accuracy requirement. The amount of data was reduced from 208 to 26. Figure 2 is a comparison of three-dimensional morphology between original data and multi-principal component characterization data in the metal tensile test.

![Three-dimensional morphology comparison between original data and first principal component characterization data in the metal tensile test](image)

In summary, the displacement data has the following characteristics after being compressed by PCA.

1. The size of the stored data is reduced, and the original data dimension is compressed from A to B. For the structural deformation, the main information of the data matrix is concentrated in the first few principal components, and the 1-3 value of R can meet the requirement that the cumulative variance interpretation rate is greater than 0.8.
2. PCA data compression has the function of noise reduction, and the processed data tends to be smoother, which can effectively eliminate the impact of measurement errors.

3.3 Model Parameter Identification Calculation

For the model parameter identification of metal material tensile test, parametric modeling is carried out in the finite element. The model size is shown in Figure 3, and set the initial material parameters, elastic modulus $E$ and Poisson’s ratio $\nu$. The boundary condition is fixed at one end and the displacement load is applied at the other end. The element type is C3D8I. The data amount is determined according to the mesh size. The data points in the full field are collected to obtain the displacement information of the metal material during the tensile process.

![Q235 steel tensile test model](image)

Selection of 208 data with large raw data and high cumulative variance interpretation rate for identification. Figure 4 shows the comparison between the direct reverse method and the data PCA compression reverse method. Figure 5 shows the iterative process of two material parameters. It can be seen that both the objective function value of the Eq. (7) and the model parameters to be sought in this paper approximate the objective function value of the classical Eq. (8) and the parameter value to be determined.
The results show that the iterative process of the direct inverse method and data PCA compression inverse method are quite different. From Figure 6, it can be seen that with the increase of data, the number of convergence iterations of the proposed method is significantly less than that of the direct inverse method. Moreover, the iterations of the proposed method do not increase significantly with the data risen. Compared with the direct inverse method, both of them have high accuracy in obtaining material parameters to be identified.
3.4 Noise Resistance

In the DIC technology measurement, the random error caused by image noise has a great influence on the normal operation of the system. It is necessary to analyze the anti-noise performance of the algorithm. This paper mainly considers the influence of random noise. According to the measurement error of Vic-3D is less than 100 μm. The experimental approach is to add Gaussian random noise with a mean value of 0 and variance of 0.0001 to the field measurement data. The effect of the direct inverse algorithm and model parameter inverse algorithm based on data PCA compression under the introduction of noise is analyzed.

<table>
<thead>
<tr>
<th>Model</th>
<th>Number of parameters to be identification</th>
<th>Raw data</th>
<th>Noise</th>
<th>Direct identification Method</th>
<th>Data PCA Compression identification Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile test</td>
<td>2</td>
<td>208</td>
<td>0.0001</td>
<td>57</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3.52%</td>
<td>0.65%</td>
</tr>
</tbody>
</table>

The experiment in Table 1 shows that the accuracy of the model parameter inverse algorithm based on PCA compression is less when the noise is introduced. At the same time, compared with the direct inversion, the iterations of the algorithm are significantly reduced with or without noise, showing faster convergence speed, stronger stability and noise suppression ability.

4. Summary

The method for model parameter identification based on data PCA compression solves the problems of a large amount of DIC measurement data and difficult selection of points. The method uses the whole field data points in the measurement plane, and selects the principal component information that satisfies the cumulative variance interpretation rate for data compression. It is important to ensure that the primary features of the DIC displacement measurement are not lost due to compression. With the increase of data, this algorithm can achieve faster convergence accuracy and less computational time compared with the original method. Besides, the influence of noise is effectively reduced in the process of data PCA compression.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (10902077, 11772228). The Author would also like to thank Tianjin University of Science & Technology for the support in 3D DIC measurements and analysis.

References

INVERSE IDENTIFICATION OF CRACK LOCATION AND LENGTH USING MODAL PARAMETERS

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Abstract

In recent years, with the aging of infrastructures, fast and accurate damage identification method is needed to implement efficient maintenance methods. Ultrasonic testing and magnetic particle testing, for instance, can identify whether the damage exists or not, or the location of the damage. To date, however, there are only a few methods to accurately identify the geometric information of the damages of structures. In this research, we aim to construct an inverse identification method to capture the damage information, based on modal parameters, i.e., natural frequencies and mode shapes of the plate.

Key words: Modal Parameter, Numerical Optimization, Genetic Optimization, Local Optimization, Inverse Problem

1. Introduction

Structural damage identification is necessary to maintain the safety of mechanical structures. The Vibration-based identification method has received increased attention over the past few decades because modal parameters can be obtained by experimental mode analysis [1] relatively easily. For instance, Friswell et al. utilized natural frequencies and eigenvectors to identify locations where the stiffness is changed in cantilever beams [2]. Taghi et al. utilized natural frequencies to identify the location and depth of a crack in cantilever beams using a genetic algorithm [3]. Furthermore, Lu et al. applied niche PSO to identify the location where the stiffness is changed in plate-like structures [4]. However, there are only a few attempts that have applied a genetic algorithm to identify cracks in plate-like structures. In this paper, an inverse identification method of crack location and depth of a plate-like structure is proposed, which utilizes the Genetic Algorithm (GA) and Sequential Quadratic Programming (SQP) for the minimization of a parameterized functional that represents a difference between measured and computed modal parameters. This paper is organized as follows. In section 2, the mathematical formulation of the crack identification method is described. In section 3, case studies are presented to demonstrate the effectiveness of the proposed algorithm. Moreover, the influence of measurement errors in modal parameters on the results of crack identification is discussed. Finally, the conclusions of the paper are drawn in section 4.

2. Mathematical Formulation

When a structure experiences a set of damages, its modal parameters go through some changes compared with those of the original structure. The changes in modal parameters are dependent on the states of the damages in the original structure. It means that there is a possibility that the changes in the modal parameters can be utilized for the detection of the damages. In other words, when the changes are known, the damages to the structure can be estimated. This process of estimating to damage states using modal parameters is an inverse problem, because the changes in the modal parameters can be obtained by solving the forward eigenvalue problem of the damaged structures. In this paper, we restrict our attention to cracking of structures as a representative example of the damages. We formulate this inverse problem of identifying the crack as the minimization of the sum of error functionals that are functions of modal parameters, as follows:

\[ \min_{s,t} \left| sE(x) + tF(x) \right| \]

\[ s + t = 1 \]
where $x$ is a vector containing the geometric information of cracks, $s$ and $t$ are the weighting coefficients to be investigated. Moreover, $E(x)$ is the average of the natural frequency errors and $F(x)$ is the cosine similarity of eigenvectors, which are defined as follows:

$$E(x) = \sum_{i=1}^{\hat{\nu}} \left[ \left( \frac{\omega_i(x) - \omega_i}{\omega_i} \right) \psi_i \right] / \hat{\nu}$$

$$F(x) = \sum_{i=1}^{\hat{\nu}} \left[ 1 - \frac{\| \phi_i(x) \| \| \psi_i \|}{\phi_i(x) \psi_i} \right] / \hat{\nu}$$

![Global Optimization using Genetic algorithm](image1)

![Local Optimization using SQP method](image2)

Figure 1. The flowchart of the combined optimization strategy

where $\omega_i(x)$ is the predicted $i$-th natural frequency of the cracked structure with assumed crack geometry, $\omega_i$ is the measured $i$-th natural frequency of the cracked structure, $\phi_i(x)$ is the predicted $i$-th eigenvector of the cracked structure with assumed cracked geometry, $\phi_i$ is the measured $i$-th eigenvector of the cracked structure. $\psi_i$'s and $\omega_i$'s are the weighting coefficients to be applied. $\hat{n}$ is the number of modes to be taken into account. The estimation of natural frequencies and their corresponding mode shapes, or eigenvectors, can be conducted by numerical methods, such as Finite Element Analysis (FEA). In this paper, the measured modal parameters have been computed also by FEA.

![Generate initial population randomly](image3)

Calculate eigenfrequencies and eigenvectors from the damaged plates

Test data $[\hat{\omega}, \hat{\phi}]$

Calculate evaluation value for each individual

Sort, and convergence check?

Crossover

Mutation

Max iteration?

Output $[\text{Crack location, Crack depth}]$

In order to minimize the objective function, we developed an algorithm where GA and SQP are utilized sequentially. We employ this strategy because the variation of the objective function in the parametric space is not monotonic, and the function may have multiple stationary points. If we would use only continuous optimization methods such as SQP, the algorithm is easily trapped by such local minima and the optimal solution cannot be found. The algorithm is
elaborated in detail with a flowchart of the algorithm in Figure 1. First, global optimization is conducted by GA to obtain a rough estimate of the solution. Second, local optimization is then conducted by SQP to obtain a more precise solution near the rough estimate of the solution found by GA. This is achieved by choosing the rough estimate obtained by GA as the initial value of SQP. The flow chart of GA is shown in Figure 2. First, the initial population is randomly generated. Second, modal parameters of the damaged plate are calculated by solving the eigenvalue problem of the structure with the crack corresponding to each individual in the algorithm. Third, comparing test data with each modal parameter, the evaluation value is calculated for each individual. After that, each individual is sorted by ascending order. If the best individual is unchanged for several generations, the algorithm is terminated and outputs the convergence value. Otherwise, crossover and mutation are conducted to avoid local minima. We set the mutation rate at 30%. Finally, the algorithm is terminated and outputs the converged value if the number of iterations exceeds the pre-defined maximum number of iterations. We have set the number of maximum iterations to 200. Otherwise, the algorithm returns to the calculation of modal parameters.

![Fixed Crack](image)

Figure 3. Cracked plate (297mm × 210mm × 1mm)

### 3. Numerical Results

#### 3.1 Finite Element Model

<table>
<thead>
<tr>
<th>$x_{[mm]}$</th>
<th>$y_{[mm]}$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>9,505</td>
<td>30,695</td>
<td>59,103</td>
<td>104,399</td>
<td>133,457</td>
</tr>
<tr>
<td>150</td>
<td>10</td>
<td>9,500</td>
<td>30,638</td>
<td>58,966</td>
<td>104,220</td>
<td>133,393</td>
</tr>
<tr>
<td>150</td>
<td>30</td>
<td>9,461</td>
<td>30,140</td>
<td>57,930</td>
<td>102,707</td>
<td>132,443</td>
</tr>
<tr>
<td>130</td>
<td>50</td>
<td>9,374</td>
<td>29,118</td>
<td>56,031</td>
<td>100,190</td>
<td>129,826</td>
</tr>
</tbody>
</table>

Figure 3 illustrates a numerical example that represents a cracked structure to be investigated. It is a fixed plate with dimension 289×210×1, 0mm, and the plate contains a crack located at $x_{[mm]}$ from the fixed end, and the depth of the crack is $y_{[mm]}$, as shown in Figure 3. The gap between the surfaces of the crack is assumed to be 1,0mm for any values of $x$ and $y$. The numerical model is constructed by using FEM, which was implemented in COMSOL multiphysics.

#### 3.2 Effects of Crack Location and Length

In order to visualize the changes in modal parameters, eigenvalue analysis has been conducted, and the results are shown in Figure 5, where Figure 5(a) shows the mode shapes of a plate without the crack, whereas Figure 5(b) shows those of the cracked plate, respectively. It is clear that the mode shapes shown in Figure 5(b) are distorted due to the crack in comparison with those of the plate without the crack. This result suggests that the crack changes the mode shapes of the plate to some extent. Moreover, Table 1 shows the changes in natural frequencies between a plate
without crack and a cracked plate. It is obvious that natural frequencies change due to the crack of the plate. In particular, the natural frequencies monotonically decrease as $y$ increases, or the crack grows. Therefore, since both natural frequencies and their corresponding mode shapes change due to the crack, if the change can be related to the crack quantitatively, the crack can be identified using the modal parameters.

Figure 4. Comparison of Mode Shapes ($i=1, \ldots, 5$)

Figure 5. Effects of crack location and depth
The proposed crack identification method is applied to plates that contain some kind of cracks. The optimization algorithm described in section 2 has been implemented by MATLAB. The conditions and results are shown in Figure 5. For the locations considered, identification errors become small as crack depth becomes deeper. This makes sense because the change of modal parameters should be small when the crack depth is shallow, i.e., the sensitivity of the objective function with respect to $y$ is small. Therefore, the algorithm cannot easily find the minimum. Moreover, identification errors are smaller for the case with the crack location, $x=150\text{mm}$, than for the other cases considered.

### 3.3 Effects of Weights

In order to consider the effects of the ratio between $E(x)$ and $F(x)$, the proposed identification method has been conducted for $[s,t]=[1.0, 0.0]$, $[0.5, 0.5]$, and $[0.0, 1.0]$. Crack location and depth have been assumed to be $[x,y]=[150, 30]$. The result of the optimization is shown in Figure 6. The identification error of the depth becomes the minimum when $[s,t]=[1.0, 0.0]$. The identification error of location becomes the minimum when $[s,t]=[0.5, 0.5]$. In summary, even though the effects of weights on the optimization accuracy cannot be well explained, the results indicate the crack identification can use both $E(x)$ and $F(x)$, i.e., natural frequency or eigenvectors.

### 3.4 Effects of Measurement Errors

In reality, modal parameters to be measured by experimental modal analysis, more or less, contain measurement errors. To demonstrate the capability of the proposed algorithm to this situation, we inject noise into computed modal parameters to mimic measurement errors, as follows,

$$\tilde{\omega}_i = (1 + \epsilon) \omega_i$$

$$\tilde{\varphi}_i = (1 + \alpha \epsilon) \varphi_i$$

where $\omega_i$ is the measured $i$-th natural frequency. $\tilde{\varphi}_i$ is the $i$-th eigenvector, $\omega_i$ is the natural frequency with the injected noise. $\tilde{\varphi}_i$ is the eigenvector with the injected noise, $\epsilon$ is a measurement error, $\alpha$ is a random variable that varies within $[-1, 1]$. First, only the functional $E(x)$ has been considered, where the errors in natural frequencies are taken into account. Crack location and depth were assumed to be $x=150\text{mm}$ and $y=30\text{mm}$, respectively. The proposed method is then applied to this situation, and the results are shown in Figure 7(a). Second, only the eigenvectors are taken into account, i.e., the functional $F(x)$ has been considered. The other conditions are the same for both cases. The results are shown in Figure 7(b). For these cases, the measurement error levels were assumed to be $\epsilon = 0.01\%, 0.03\%, 0.05\%, 0.07\%, \text{and} 0.09\%$ in natural frequency, $\epsilon = 1.0\%, 2.0\%, 3.0\%, 4.0\%$ in eigenvector. In the case of natural frequency, as measurement errors increase, crack location and depth errors increase. Measurement error greatly increases from $\epsilon = 0.07\%$ to $\epsilon = 0.09\%$. In the case of eigenvector, identification error monotonically increases as $\epsilon$ increases, except for $\epsilon = 2.0\%$. Moreover, in comparison with the natural frequency, identification error is not large even when the measurement error is $\epsilon = 4.0\%$ for the mode shapes.
4. Conclusions

In this paper, a method to solve an inverse problem of finding the length and depth of cracks in plate-like structures has been developed. The algorithm combines GA and SQP to avoid local minima yet to achieve high accuracy using local optimization method. The Mathematical formulation is described and applied to a representative example with a cracked plate. First, we visualized that the shape of damage affects natural frequencies and their mode shapes. As a result, as crack depth becomes deeper, identification error becomes larger. Second, we examined the effects of weights between the functional related to natural frequency errors and eigenvector errors. Even though it is not clear yet which functional is superior in terms of the inverse problem perspective, it is shown that at least both functional can be utilized. Third, we examined the effects of injecting error into modal parameters. As measurement error becomes large, the identification error becomes large.

References

STATIC AEROELASTIC EFFECTS STUDY ON AERODYNAMIC PERFORMANCE OF LARGE TRANSPORT AIRCRAFT WINGS

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Abstract

In order to meet the great improvement of large transport aircraft design requirements, the designers must break the disciplinary boundaries in traditional flight vehicle design and take advantage of multidisciplinary coupling design methods. When it comes to a large transport aircraft, the application of multidisciplinary design optimization should not ignore the static aeroelastic effects. To study these effects thoroughly, the paper establishes a high-fidelity coupled aerostructural analysis method firstly and validates its accuracy with DLR-F6 wing-body configuration. Then the aerodynamic performance of supercritical wing for a dual-aisle large transport aircraft under different flight conditions with different aeroelastic deformation conditions are evaluated by the proposed method. The influence on characteristics of cruise condition, drag-divergence conditions, buffet conditions and low-speed conditions are studied according to these results. Finally, the study shows that: the laws of aerodynamic property changes in all aspects of the configuration may show great difference or even an opposite trend when considering static aeroelastic effects. To fulfill the actual engineering requirements, the designers should consider these effects in the early phase.

Key words: Large Transport Aircraft, Static Aeroelastic Effect, Coupled Aerostructural Analysis, Multidisciplinary Optimization, Supercritical Wing

1. Introduction

As a representative of the highest level in the field of civil aviation technology in the world, the wide-body airliners, which mean the dual-aisle large transport aircraft, such as Boeing 787 and Airbus 350XWB, have shown advantages of higher economy, better comfort, longer range and lower pollution than traditional passenger aircrafts like Boeing 737. Take the aerodynamic performance as an example, the lift-to-drag ratio (L/D) of traditional aircraft, Boeing 737, is lower than 17, while that of wide-body airliners, Boeing 787, is more than 20. This huge change of performance makes it difficult for traditional Aerodynamic Shape Optimization (ASO) methods to fulfill design requirements. To solve the problem, it’s necessary to break disciplinary boundaries and take advantage of multidisciplinary coupling design methods.

When we take Multidisciplinary Design Optimization (MDO) methods into consideration, the static aeroelastic effects should not be ignored. Large transport airplanes often use the supercritical wings that have high aspect ratio and large sweepback angle, which will result in non-negligible wing deflection deformation at cruise conditions. Sometimes, the deformation of wingtip could be in meters, and the aerodynamic performance of aircraft may be greatly affected. Academia has done a lot of research work on this. For example, Piperno et al [1] have done a preliminary aerostructural optimization of a large business jet, Martins and his group [2-3] made a series of multipoint aerostructural optimization for the Common Research Model (CRM) configuration. Xin Liang [4] has adopted a loose model to carry out aerostructural optimization of a civil airliner. However, the main attention is paid to optimization methods, few work focuses on the influence of static aeroelasticity on the aerodynamic performance of large transport aircraft wings.

To study the influence thoroughly, firstly the high-fidelity closely coupled aerostructural numerical analysis method

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base on Computational Fluid Dynamics (CFD) and Computational Structural Mechanics (CSM) is established. Then the aerodynamic performance of the supercritical wing in a wide-body aircraft under different flow conditions with different aeroelastic deformation conditions is evaluated by the proposed method. The influence of aeroelastic deformation on aerodynamic characteristics, such as pressure distribution, span-wise lift distribution, drag divergence Mach number \((M_d)\), buffet boundary and so on, is studied according to these results. Finally, the paper gives some guiding principles on the ASO design of the flexible wing, which will lay a good foundation for aerostructural design of large transport aircraft wings.

2. Closely Coupled Aerostructural Analysis Methodology

2.1 Aerostructural Numerical Analysis

The aerodynamic analysis uses the Reynolds-Averaged Navier-Stokes (RANS) equations in steady modes

\[
\frac{\partial U}{\partial t} + \frac{\partial F_i}{\partial x_i} = \frac{\partial G_i}{\partial t} + \frac{\partial G_i}{\partial x_i} = 0
\]

Where \(U\) is conservative variables, \(F_i, F_j, F_k\) is non-viscous fluxes, \(G_i, G_j, G_k\) is viscous fluxes. The solver uses structured, multi-block mesh and the Spalart-Allmaras (SA) turbulence model. The central discretization scheme with artificial dissipation is used. More details about CFD solver are listed by Chen Song et al.\cite{5}.

The structural displacements \(u\) are solved by linear shell finite elements. For linear analysis, the structural discipline residuals are

\[
s(u) = Ku - F = 0
\]

Where \(K\) is the structure stiffness matrix and \(F\) is the load vector.

Figure 1 shows the schematic of the rigid link method and iteration model. The paper uses a rigid link method\cite{6} as a CFD/CSM transfer scheme. These rigid links are constructed between the aerodynamic surface mesh and the structural surface mesh. And they provide an easy way to exchange data, which enforces both displacement continuity and energy conservation across the multidisciplinary interface. The closely coupled model is used in aerostructural iteration computation to improve the efficiency and accuracy of the solution.

Finally, the mesh deformation algorithm used in this paper is based on the Inverse Distance Weighting (IDW) interpolation method\cite{7}. The method is robust and fast enough to deal with wing deformation in the aerostructural simulation.

![Figure 1. Schematic of rigid link method (left) and closely coupled model (right)](image)

2.2 Numerical Solution Validation

The proposed method is validated by using the DLR-F6 wing-body configuration, as shown in Figure 2. The CFD grid is provided by Second AIAA CFD Drag Prediction Workshop (DPW II), with 5.8 million cells. The wingbox is composed of an upper and lower skin, 3 spars and 31 ribs. The flow condition is

\[
Ma = 0.75, Re = 3 \times 10^6
\]

Comparing numerical results with wind tunnel data at NASA’s National Transonic Facility (NTF)\cite{9} and the simulation results provided by Keye and Rudnik\cite{10}, the span-wise wing bending distributions for different coefficients of lift \((C_L)\) show little difference, which validates the method’s accuracy.
3. Static Aeroelastic Effects Study

By using the closely coupled aerostructural analysis method, we now evaluated the aerodynamic performance of the supercritical wing in a wide-body aircraft under different flow conditions with different aerelastic deformation conditions. The influence of aerelastic deformation on aerodynamic characteristics, such as pressure distribution, span-wise lift distribution, drag divergence Mach number, buffet boundary and so on, is studied according to these results.

3.1 Aerodynamic Model

The basic aerodynamic model comes from a gradient-based ASO of Common Research Model (CRM) configuration [11] at cruise condition which is

\[ M_\infty = 0.85 \quad C_L = 0.48 \quad H = 11.5 \text{km}. \]

The objective of the optimization is to reduce the coefficient of drag \((C_d)\) while the coefficient of moment \((C_m)\) should be kept to zero. More details about the optimization system are provided by Chen Song et al [5]. Finally, we get the aerodynamic model, as shown in Figure 3, whose wing leading edge sweepback angle is \(35^\circ\) and aspect ratio is approximately 10.5. The CFD grid has 12 million cells. The aerodynamic numerical results will be shown in the following sections.

3.2 Structural Model

A simplified structural model is adopted from the reference wingbox model provided by the CRM website [12]. As shown in Figure 4, the wingbox includes an upper and lower skin, 3 spars, 26 stringers and 50 ribs. The center wingbox section has 4 ribs orientally parallel to the fuselage, with the remaining 46 ribs distributed along the span of the wing perpendicular to the leading edge, with the exception of the closeout rib at the tip. All components in the wingbox structure are modeled by using aluminum 2024 series alloy and constructed with 93592 quadrilateral finite elements. So by scaling thickness of the whole model, the aerostructural numerical analysis results will have different wing deformation.

Figure 4 also shows the structural constraints under consideration. To keep the same effects with actual wing clamping, the inner most wingbox rib is fixed at the symmetry plane, so all the displacements and rotations for the elements in that rib are limited to zero. In addition, the rib at the wing-fuselage junction is constrained such that the vertical and longitudinal displacements are fixed.
3. 3 Flow Conditions

The well-designed aircraft should not only show good design performance, but also yield robust off-design performance. And it is necessary to consider multipoint design to ensure good performance for a range of flight conditions. So to study static aeroelastic effects on aerodynamic performance, the paper chooses some key flight conditions including cruise condition, drag-divergence conditions, buffet conditions and low-speed conditions, as shown in Table 1.

<table>
<thead>
<tr>
<th>Flow Conditions</th>
<th>Altitude / km</th>
<th>Mach Number</th>
<th>Angle of Attack</th>
<th>$C_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cruise Condition</td>
<td>11.5</td>
<td>0.85</td>
<td>/</td>
<td>0.48</td>
</tr>
<tr>
<td>Drag-Divergence Conditions</td>
<td>11.5</td>
<td>0.86, 0.87</td>
<td>/</td>
<td>0.48</td>
</tr>
<tr>
<td>Buffet Conditions</td>
<td>11.5</td>
<td>0.85</td>
<td>0°, 2°, 4°, 5°, 8°</td>
<td>0.48×1.3 = 0.624</td>
</tr>
<tr>
<td>Low-Speed Conditions</td>
<td>0</td>
<td>0.2</td>
<td>11°, 12° to stall</td>
<td>/</td>
</tr>
</tbody>
</table>

3. 4 Influence on Performance of Cruise Condition

The overall stiffness is changed by setting the thickness of the structure model with 20 mm, 30 mm, 40 mm, 50 mm respectively. In this way, the configuration can be evaluated under different deformation conditions. Table 2 shows the weight and deflection deformation of the wingtip trailing edge with different thickness. Figure 5 also shows different wing deformation. The numerical results are consistent with the theoretical expectation that the wing deformation decreases with the growth of structure thickness.

<table>
<thead>
<tr>
<th>Thickness / mm</th>
<th>Mass / kg</th>
<th>Deflection Deformation of Wingtip Trailing Edge / m</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>28501.05</td>
<td>1.227</td>
</tr>
<tr>
<td>30</td>
<td>39751.57</td>
<td>0.790</td>
</tr>
<tr>
<td>40</td>
<td>53002.10</td>
<td>0.549</td>
</tr>
<tr>
<td>50</td>
<td>66252.62</td>
<td>0.395</td>
</tr>
</tbody>
</table>

Figure 5. Wing deformation with different thicknesses
For a well-designed wing of transport aircraft, the aerodynamic shape at cruise condition will deviate from the optimal and its performance will deteriorate when considering static aeroelastic effects. As results shown in Table 3, the coefficients of drag and the coefficients of moment have increased, which will worsen cruise performance. The results in Table 4 show the relative change of aerostructural analysis results compared with aerodynamic analysis results. When the deflection deformation of the wingtip trailing edge is 1.23 m, the lift-drag performance has lost by 4.53%. The paper has selected 4 span-wise positions to show the difference of coefficients of pressure ($C_p$) distribution and airfoil geometry, as shown in Figure 6. The wing deformation is mainly reflected in the change of the deflection and geometric twist. As the reduction of structure stiffness, the deflection increases while the twist near the outer wing decreases. And it eventually leads to the aeroelastic effects of reducing the tip load, which can be seen in the comparison of span-wise lift distribution as shown in the right Figure at Figure 6. From the comparison of $C_p$ distribution, the study finds that the strength of shock at each position increases, the shock position is delayed. And there will be a new shock area in the wing surface.

Table 3. Influence of Different Deformation on Cruise Performance

<table>
<thead>
<tr>
<th>Thickness / mm</th>
<th>Angle of Attack</th>
<th>$C_l$</th>
<th>$C_{d, lift}$ counts</th>
<th>$C_m$</th>
<th>$L/D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>3.03025°</td>
<td>0.48</td>
<td>258.71</td>
<td>0.038890</td>
<td>18.55</td>
</tr>
<tr>
<td>30</td>
<td>2.83545°</td>
<td>0.48</td>
<td>254.05</td>
<td>0.025935</td>
<td>18.89</td>
</tr>
<tr>
<td>40</td>
<td>2.72820°</td>
<td>0.48</td>
<td>251.71</td>
<td>0.018449</td>
<td>19.07</td>
</tr>
<tr>
<td>50</td>
<td>2.66007°</td>
<td>0.48</td>
<td>250.23</td>
<td>0.013509</td>
<td>19.18</td>
</tr>
<tr>
<td>$+\infty$ (Aerodynamic Analysis)</td>
<td>2.48926°</td>
<td>0.48</td>
<td>247.01</td>
<td>0.000242</td>
<td>19.43</td>
</tr>
</tbody>
</table>

Table 4. Influence of Different Deformation on Cruise Performance (relative change)

<table>
<thead>
<tr>
<th>Thickness / mm</th>
<th>Angle of Attack</th>
<th>$C_l$</th>
<th>$C_{d, lift}$ counts</th>
<th>$C_m$</th>
<th>$L/D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>+21.73%</td>
<td>0.00%</td>
<td>+4.74%</td>
<td>+15970, 25%</td>
<td>-4.53%</td>
</tr>
<tr>
<td>30</td>
<td>+13.91%</td>
<td>0.00%</td>
<td>+2.85%</td>
<td>+10616, 94%</td>
<td>-2.78%</td>
</tr>
<tr>
<td>40</td>
<td>+9.60%</td>
<td>0.00%</td>
<td>+1.90%</td>
<td>+7523, 55%</td>
<td>-1.85%</td>
</tr>
<tr>
<td>50</td>
<td>+6.86%</td>
<td>0.00%</td>
<td>+1.30%</td>
<td>+5482, 23%</td>
<td>-1.29%</td>
</tr>
<tr>
<td>$+\infty$ (Aerodynamic Analysis)</td>
<td>2.48926</td>
<td>0.48</td>
<td>247.01</td>
<td>0.000242</td>
<td>19.43</td>
</tr>
</tbody>
</table>

Figure 6. Comparison of aerodynamic performance at cruise condition

3.5 Influence on Performance of Drag-Divergence Conditions

$M_{sl}$ is the Mach number at which the aerodynamic drag begins to increase rapidly as the Mach number continues to increase. $M_{sl}$ is an important criterion of aerodynamic performance. The larger $M_{sl}$ is, the more robust the off-design performance of the aircraft will be.
Table 5 shows the $C_D$ increments with Mach number. The drag- divergence performance may be better in some aerostructural simulation results compared with aerodynamic analysis results. However, with the reduction of structure stiffness, $C_D$ increments become larger and larger, which will decrease the $M_{cl}$.

<table>
<thead>
<tr>
<th>Thickness / mm</th>
<th>$C_{D_{0.45}} - C_{D_{0.95}}$ / counts</th>
<th>$C_{D_{0.67}} - C_{D_{0.84}}$ / counts</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>15.65</td>
<td>33.13</td>
</tr>
<tr>
<td>30</td>
<td>14.36</td>
<td>31.96</td>
</tr>
<tr>
<td>40</td>
<td>13.60</td>
<td>31.66</td>
</tr>
<tr>
<td>50</td>
<td>13.29</td>
<td>31.75</td>
</tr>
</tbody>
</table>

$+\infty$ (Aerodynamic Analysis)

$\chi = -V \times V_{\text{frestream}}$

3.6 Influence on Performance of Buffet Conditions

The behavior of the configuration near buffet conditions is another critical aspect of transonic wing design, which is not usually considered in ASO. As the Mach number or $C_L$ increases, shock waves gradually increase in strength, which will interact with the boundary layer and eventually lead to unsteady flow separation and oscillation of the shocks. Federal Aviation Administration (FAR) and Civil Aviation Authority (CAA) regulations stipulate that commercial transport aircraft must maintain at least 1.3g margin from the cruise condition to buffet onset. That is why the paper chooses the 1.3 times cruise $C_L$ as one of the evaluation conditions in Table 1.

The physical mechanism for buffet onset is the growth of shock-induced flow separation. To quantify the separated flow, the paper uses a separation sensor function [15] as an evaluation criterion, which is shown as follows:

$$
\chi = -V \times V_{\text{frestream}}
$$

In this equation, $A_{ref}$ is wing reference area, $S$ is the wing surface, $V$ is velocities at the cell center immediately adjacent to the wall and $V_{\text{frestream}}$ is velocities of the free stream at far field. $k$ and $\lambda$ are free parameters, where $k$ determines the sharpness of function curves, and $\lambda$ can be used to shift the function curves to the left or right.

The separation sensors at buffet conditions for $\lambda = -0.1$ and $k = 10$ are shown in Figure 7. By taking static aeroelastic effects into consideration, the area of separation flow reduces to some extent in each state with the change of angle of attack. The separation region decreases with the reduction of the structure thickness. The ASO of basic configuration did not consider buffet performance, which may be the reason why aerodynamic analysis results have the highest value of separation sensors. On the other hand, the aeroelastic effects of reducing the tip twist will reduce area of flow separation at outer wing, which is beneficial to expand the buffet boundary.

![Figure 7. Comparison of separation sensors](image1)

![Figure 8. Comparison of stall performance at low-speed conditions](image2)

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3.7 Influence on Performance of Low-Speed Conditions

For the same reason, the decrease of twist at the outer wing will cause $C_l$ loss at the same angle of attack and delay the stall angle at low-speed conditions. From the aerostructural analysis results shown in Figure 8, we can find that, with the growth of thickness, $C_l$ at low-speed conditions show an increasing trend. According to this rule, in the case of infinite thickness, that is, without considering static aeroelastic deformation, the wing separation area should be the largest. However, the aerodynamic simulation shows the opposite results, which has the least separated flow. That may be because the basic configuration has better performance of suppressing separation before wing deformation.

What’s more, the study of influence of low-speed performance indicates that, due to the large deformation of the outer wing, the variation of stall performance and wing separation characteristics considering the static aeroelastic effects are no longer consistent with the rule without considering these deformations. To get a well-designed scheme of the supercritical wing, it is important for the designers to consider the static aeroelastic effects in the early phase.

4. Conclusions

MDO methods have a great significance that will supply the wide-body airliners with advantages of higher economy, better comfort, longer range and lower pollution [14]. In this paper, by using the closely-coupled aerostructural analysis method, we evaluated the aerodynamic performance of the supercritical wing in a wide-body aircraft under different flow conditions with different aeroelastic deformation conditions. The influence on aerodynamic characteristics of cruise condition, drag-divergence conditions, buffet conditions and low-speed conditions are studied according to these results.

For a well-designed wing of transport aircraft, the aerodynamic shape at cruise condition will deviate from the optimal and its performance will be worsening when considering static aeroelastic effects. As the results shown in the paper, when the deflection deformation of wingtip trailing edge is 1.23 m, the lift-drag performance has lost by 4.53%. The coefficients of drag in the whole flow conditions have increased. The drag-divergence performance may be better in some cases. However, with the reduction of structure stiffness, wing deformation becomes larger and larger, which will badly decrease the $M_{cd}$. The aeroelastic effects of reducing the tip load will reduce area of flow separation at outer wing, which are beneficial to expand the buffet boundary. For the same reason, the decrease of twist at the outer wing will cause $C_l$ loss at the same angle of attack and delay the stall angle at low-speed conditions.

Finally, these results show that the laws of aerodynamic property changes in all aspects of the configuration may show great difference or even an opposite trend when considering static aeroelastic effects. To fulfill the actual engineering requirements, it is necessary to consider static aeroelastic effects in the early design phase.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 11702284), Aeronautical Science Foundation of China (Grant No. 20161453011) and China COMAC Shanghai Aircraft Design and Research Institute which is gratefully acknowledged by the authors.

References

COUPLED DESIGN OF A SUPersonic engine AND
THERMAL SYSTEM

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Abstract

Supersonic aircraft are inherently challenging to design because they are highly coupled multidisciplinary systems that operate in a diverse range of flight regimes. As modern supersonic aircraft become lighter and require greater electrical capabilities, the propulsion and thermal management systems become performance critical. In this work, we develop a coupled propulsion-thermal model and perform gradient-based optimization to simultaneously design the engine and thermal system. After constructing the full multidisciplinary system, we solve a series of optimization problems using the coupled model with both “lifting” (refrigerating) and non-lifting thermal management systems. We sweep through a range of heat transfer requirements to evaluate how the performance of the optimal engine varies. Depending on heat transfers requirement, optimization reduces fuel consumption by up to 2.6% when using a thermal lifting system, showing the relative importance of considering the engine and thermal systems simultaneously.

Key words: Multidisciplinary Design Optimization, Engine Design, Thermal Systems, Gradient-Based Optimization

1. Introduction

The next generation of aircraft, in both the commercial and military sectors, must meet increasingly stringent requirements, including constraints on fuel consumption, noise, emissions, heat, and cost. To meet these performance metrics, aircraft designers must innovate, from the system level to individual parts. The propulsion system has a particularly large effect on overall aircraft performance, since it both propels the aircraft and provides auxiliary power.

As aircraft become lighter and more electrified, thermal constraints become more limiting and critical for the system level design. Military aircraft are especially constrained thermally due to radar, communications, and weapons systems. Many operational constraints, such as radar observability and weight limits, reduce the available ways to dissipate heat. Heat can be rejected via airstreams within the engine without drastically increasing radar signature. Because of this, interactions between the propulsion and thermal systems have been studied extensively for military aircraft [1, 2-3, 11, 19]. The effect of heat exchangers within engine bypasses ducts is a specific area of interest. Bypass heat exchangers are useful in both transonic and supersonic aircraft. Prior work has studied to bypass heat exchangers using wind-tunnel experiments [5, 14] and computational methods [1, 5, 14, 19]. Bypass airflow heat exchangers are also useful in transonic commercial airlines, including the Boeing 787, which has a surface cooler to reject engine oil heat and generator heat [6], showing the wide applicability of this research.

Instead of conducting experiments to evaluate new aircraft system designs, we use computational modeling, which allows designers to rapidly evaluate systems at a relatively low cost. To capture the effects of bypass heat exchangers, we need to model both the engine and thermal systems simultaneously and in a fully coupled manner. This has been done previously by Allison et al. [1] and by Puterbaugh et al. [19], who found interesting trade-offs between engine performance and thermal dissipation capacity. However, few of these studies looked at changing the design of the engine or thermal system, and no previous study has performed optimization on the coupled system.

By using Multidisciplinary Design Optimization (MDO) [18], we can vary the design of a complex system to minimize an objective function subject to constraints. However, to explore a large design space efficiently, we need to
use gradient-based design optimization [16]. Gradient-based optimization methods require the total derivatives of the system for all functions of interest. Generally, this means that the multidisciplinary computational models must be developed with efficient derivative computation in mind. In the case of coupled engine-thermal systems, we need the derivatives of any quantity of interest (fuel burn, thrust produced, heat dissipated) with respect to all engine and thermal design variables.

In this paper, we first discuss the relevant theory behind the engine and thermal system modeling. We then use NASA’s OpenMDAO framework [7] to construct a model that couples the engine and thermal system and efficiently computes the relevant gradients. We construct a 3-stream, split-flow supersonic turbofan engine and examine different thermal system architectures, including a simple heat-lifting system. We then perform a series of optimizations to show how the engine and thermal systems interact together and find optimal designs for a variety of thermal loads and flight conditions. We find that the addition of the thermal lifting system decreases the fuel consumption for a thermally-constrained supersonic flight condition by amounts ranging from 1.5% for 2.5 kW of heat rejection, to 2.6% for 15 kW of heat rejection.

2. Model Description

2.1 Propulsion Model

The propulsion system is modeled using a 1D thermodynamic cycle implemented in pyCycle [8], which is developed using OpenMDAO [7]. pyCycle is designed to model propulsion systems within the context of larger systems, especially for MDO applications. The tool provides a library of elements, such as inlets, compressors, combustors, and turbines, that can be combined to create an engine cycle. pyCycle has similar modeling capabilities as the industry standard, NPSS [12]. pyCycle also provides efficient analytic derivatives, which are necessary for gradient-based optimization.

In this work, we use pyCycle to construct our three-stream split-flow supersonic engine model, which is shown in Figure 1. We also show the engine model with the addition of the thermal lifting system in Figure 2, which is explained in detail in section 2.2.2. Throughout the paper, we perform single-point design optimization, though pyCycle can run multipoint optimization problems in parallel. We use a supersonic top-of-climb flight condition throughout this work, since a previous multipoint design study found it was the most limiting condition [10]. We include installation effects using a series of surrogate models based on data from a fixed-shape supersonic inlet. Based on the free stream Mach number, projected capture area, and inlet area, we compute bleed, bypass, and spillage drag terms to account for installation losses. These drag contributions are subtracted from the net thrust of the rest of the engine cycle to obtain an installed net thrust, which is used to measure engine performance. Split-flow turbofans are realistically not used in the supersonic regime, though we use a split-flow architecture here to study engine-thermal interactions with a relatively simple engine model. For the same reason, the engine architecture used here does not have an afterburner.

![Figure 1. The 3-stream engine schematic with bypass duct heat exchanger](image-url)
2. 2 Thermal System Model

2.2.1 Heat Exchanger Description

Heat exchangers provide an interface with favorable heat transfer properties between hot and cold fluids. The steady-state heat transfer of a heat exchanger can be computed via the Nusselt-effectiveness method as

\[ q = \epsilon \frac{UA_{\text{overall}}}{N_{\text{RT}}} (T_{\text{w,in}} - T_{\text{w,out}}) \]  

(1)

\[ N_{\text{RT}} = \frac{UA_{\text{overall}}}{C_{\text{min}}} \]  

(2)

\[ \epsilon = \Phi \left( N_{\text{RT}}, C_{\text{min}} \right) \]  

(3)

where \( q \) is the heat transfer rate, \( \epsilon \) is the heat transfer effectiveness, \( T_{\text{w,in}} \) are the fluid inlet temperatures, \( N_{\text{RT}} \) is the number of thermal units, \( UA_{\text{overall}} \) is the overall heat transfer coefficient times the corresponding heat transfer area, \( C_{\text{min}} \) and \( C_{\text{max}} \) the maximum and minimum values of the fluid heat transfer capacity \( \dot{m}c_p \) for the hot and cold sides, and \( \Phi \) is an analytical or empirical function that depends on the flow arrangement of the heat exchanger (for example, crossflow) [13].

For this study, we modeled crossflow, plate-fin heat exchangers with offset strip fin geometry. This configuration is considered a “compact” heat exchanger because it has a large surface area to volume ratio, suitable for gas-liquid or gas-gas exchange in weight-sensitive aerospace applications [13]. The offset strip fin geometry improves convective heat transfer coefficients compared to straight fins. The heat exchanger design problem involves choosing appropriate geometry and material thickness to satisfy heat transfer, pressure loss, weight, and volume requirements. Figure 3 illustrates a cross section of plate-fin flow channels, including the parameters width, height, fin thickness, and plate thickness.

We used a heat exchanger model from the OpenConcept\(^1\) library. The OpenConcept is an aircraft “conceptual design and optimization toolkit” built on the OpenMDAO framework, including simple, conceptual-level models of systems components [4]. The OpenConcept [4] heat exchanger models implements the \( N_{\text{RT}}-\epsilon \) equations to compute heat transfer and uses an empirical correlation from Manglik and Bergles [17] for convective heat transfer and friction coefficients specific to the offset strip fin geometry. For this study, we used values representative of an air-liquid heat exchanger, with cold-side channel width and height 1 mm, and hot-side channel width 14 mm by 1.35 mm. We did not vary the heat exchanger geometry design variables during optimization but optimizing these parameters would most likely improve system performance slightly. The heat exchanger in the bypass stream of the engine occupies the entire duct. We achieve this by setting the frontal area of the heat exchanger to be same as the bypass duct area computed by the engine design.

\(^1\) https://github.com/mdolah/openconcept/
2.2.2 Lifting System Description

The ability for the heat exchanger to transfer thermal energy into the bypass stream is dependent on its surface area, the fluid properties at both the hot and cold sides of the heat exchanger, as well as the temperature difference between the two fluids. One way to increase the temperature difference is to use an Air Cycle Machine (ACM), a type of heat pumps which uses mechanical work to transfer heat from a cooler fluid to a warmer fluid. Although more total heat is produced due to thermodynamic cycle inefficiency, the heat exchanger transfers heat faster due to the larger temperature differential. To power this thermal lifting system, energy can be used from bleeding air or shaft power off-takes from the engine.

In this study, we use shaft power from the high-pressure shaft to lift the thermal energy. This directly introduces a weak coupling between engine performance and thermal system efficiency as some energy must come off of the shaft to power the lifting system. However, the magnitude of the shaft power used in the lifting system is dwarfed by the total power in the engine, so this coupling is inconsequential. For a design later presented in this paper, the lifting work taken off the shaft is about 33 hp, while the shaft is producing more than 19,000 hp, which means that the lifting system is using about 0.17% of the shaft power. The more important trade-off is between the amount of lifting work done and the heat generated during the process. Although the heat load is at a higher temperature as it is lifted, the greater total heat amounts that need to be dissipated influences the optimal engine design.

Nominally, this ACM operates as a closed Brayton cycle, where the working fluid that transmits heat is recirculated through a series of elements. In a closed Brayton cycle, the working fluid flows through a low-temperature heat exchanger and accepts heat. The fluid then enters a compressor and contracts, leading to a higher temperature working fluid. This heated fluid then flows through a high-temperature heat exchanger, where energy is dissipated. The fluid then goes through a turbine and expands before interacting with the low-temperature heat exchanger as the cycle continues. In our model, the low-temperature heat exchanger interfaces with the electronics load on the hot side and the Brayton cycle on the cold side. The high-temperature heat exchanger takes heat from the Brayton cycle on the hot side and transfers it to the bypass duct airstream. This realistically accounts for the losses in heat transfer between the fluids, since we do not assume instantaneous and lossless heat transfer in the lifting system.

Instead of explicitly modeling the Brayton cycle using pyCycle, we model it using a system of equations based on assumed friction losses and performance efficiencies to capture the relevant physics without adding unnecessary complexity to the model. These equations are formulated to take in the amount of lifting work provided by the engine shaft, and compute the heat loads that must be transferred at both the electronics and duct heat exchangers. We solve for the heat load from the lifting system that must be dissipated using the duct heat exchanger, and get

\[
Q = \frac{W'}{1 - \frac{T}{T_h}} = \frac{\eta_p \cdot \eta_f}{1 - \frac{T}{T_h}}
\]  

where \(Q\) is the lifted heat load, \(W'\) is the efficiency-adjusted work, \(T\) and \(T_h\) are the temperatures of the cooling fluid at the electronics and duct heat exchangers respectively, \(W\) is the work coming off the shaft, \(\eta_p\) is the shaft power transfer efficiency, and \(\eta_f\) is the friction loss efficiency. We can then solve for the cold-side heat load and get

\[
Q = Q \cdot (2 - \frac{\eta_f}{\eta_p}) W
\]

\(Q\) is the amount of heat transferred from the actual heat load, e.g., our electronics heat output. The losses due to friction are lost to the cold reservoir, leading to a higher \(Q\) value than in an ideal case. Throughout this work, we use \(\eta_f = 0.95\) and \(\eta_p = 0.95\) as nominal representative values. A schematic of the work and heat flow for this simplified cycle is shown in Figure 4. The full engine-thermal model with the lifting system included was previously presented in Figure 2, which shows the shaft power off-take that is used to power the lifting system.
2.3 Fully Coupled Model

By directly coupling the propulsion and thermal system models, we can evaluate the performance of the full system while correctly accounting for the systems’ interactions. An extended design structure matrix (XDSM) diagrams [15] showing the data passing between the models and optimizer is shown in Figure 5. The optimizer provides design variable values to the engine and thermal systems, PyCycle computes the bypass duct airflow properties, including air temperature, heat capacity, density, thermal conductivity. These properties are then passed to the heat exchanger model, which then computes the heat transfer and pressure drop across the heat exchanger to pass back to the engine model. These coupled systems are converged simultaneously using a Newton solver with an Armijo-Goldstein linesearch. Because we constructed the models in OpenMDAO with MDO in mind, we can modularly change the system grouping and solver settings based on knowledge of how best to converge the complex system.

One major advantage of using OpenMDAO [7] is that the framework automatically computes the coupled adjoint for a multidisciplinary system, which efficiently computes the total derivatives needed for gradient-based optimization. The creator of the multidisciplinary model only needs to provide the analytic partial derivatives of each component within subsystems, and OpenMDAO uses chain rule and the modular analysis and unified derivatives theory [9] to compute the total derivatives of the system. This greatly decreases the implementation cost of developing novel multidisciplinary systems that provide analytic derivatives. For this work, we compute the analytic partial derivatives of the computational blocks in the propulsion and thermal systems individually, provide these to OpenMDAO, which then computes the total derivatives for all functions of interest with respect to the design variables while accounting for the direct coupling between the disciplines.

Throughout this work, we are interested in examining the trends and coupling between the engine and thermal systems instead of the absolute numbers from the analyses and optimizations. We assume a constant temperature and single-source heat load, when in reality this thermal system would have multiple heat loads at different qualities. The interactions between the disciplines are captured here without using exact efficiencies, element maps, and sizing parameters from engine manufacturers.
3. Parameter Studies

We can use the fully coupled model to explore multidisciplinary trade-offs of interest to both engine and thermal system designers. We can vary engine design parameters, such as bypass ratios, design Mach numbers, pressure ratios, element maps, and efficiencies. On the thermal system side, we can control the heat exchanger geometric variables, such as fin length, width, spacing, and offset distance. For the thermal lifting system, we can vary the mass flow through the ACM and the heat exchanger variables that interface with the ACM. In this work, we examine a small subset of these design parameters while holding the others constant to highlight how these variations affect overall system performance.

To gain an intuitive understanding of how the thermal lifting process affects optimal engine performance, we analyzed the coupled model across a sweep of lifting work values. We used the same fixed engine design used in the BPR study and did not perform any design optimization here. Figure 6 shows the temperatures and heat loads for the hot and cold sides of the lifting system for 11 different nominal lifting work values. As the amount of lifting work done increases, the temperature difference in the ACM increases, which means the heat exchangers can dissipate more thermal energy due to the larger temperature difference between the hot and cold sides. However, the work used to lift the thermal energy must also be dissipated by the heat exchangers, which means more thermal energy is put into the bypass airstream. The optimal amount of lifting work, accounting for the increased temperature difference and larger thermal loads, depends on the bypass airflow properties and heat exchanger designs. Depending on the vehicle flight conditions, different lifting work amounts will be optimal based on the amount of heat dissipation required.

![Diagram](image)

Figure 6. As more lifting work is added to the thermal lifting system, the amount of heat that must be dissipated and the temperature of the air within the ACM increase. $Q_h$ is greater than $Q_c$ due to the added heat from the lifting work.

4. Optimization Results

As discussed previously, there are a large number of potential optimization problems that could be solved to study the design space of this model. We first compare optimal engine performance for the non-lifting and lifting thermal systems across a range of heat transfer requirements at a single flight condition. Table 1 shows the optimization problem formulations for both thermal system types. The lifting work design variable is only present in the engine-thermal model that includes the lifting system.

We are minimizing the fuel flow to the combustor while meeting a heat transfer requirement at a supersonic flight condition. The optimizer has control over fan pressure ratio (FPR), overall pressure ratio (OPR), the main BPR, and the design sea level static thrust for all cases, as well as the lifting work for the lifting system cases. The $\text{Ac}_{\text{as}}$
variable controls the capture area of the inlet, which we constrain to be equal to the capture area of the supersonic case to accurately compute installation drag. We sweep through heat transfer requirements from 2.5 kW to 15 kW for both thermal system architectures and perform optimizations at each point, prescribing the heat transfer amount using an equality constraint. For these optimizations, we hold the secondary BPR fixed to 1.0 to simplify the optimization problem.

<table>
<thead>
<tr>
<th>Category</th>
<th>Name</th>
<th>Lower</th>
<th>Upper</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective</td>
<td>Fuel flow</td>
<td>-</td>
<td>-</td>
<td>lbm/s</td>
</tr>
<tr>
<td>Variables</td>
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<td>5.0</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>OPR</td>
<td>15.</td>
<td>35.</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>( F_{\text{opt-SL}} )</td>
<td>22,000</td>
<td>35,000</td>
<td>lbf</td>
</tr>
<tr>
<td></td>
<td>( A_{\text{c-c}} )</td>
<td>600</td>
<td>1200</td>
<td>in(^2)</td>
</tr>
<tr>
<td></td>
<td>Main BPR</td>
<td>0.05</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Lifting work</td>
<td>0.5</td>
<td>-</td>
<td>kW</td>
</tr>
<tr>
<td>Constraints</td>
<td>Capture area ratio</td>
<td>1.</td>
<td>1.</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>( F_{\text{opt-TTC}} )</td>
<td>7600</td>
<td>-</td>
<td>lbf</td>
</tr>
<tr>
<td></td>
<td>Heat transfer</td>
<td>2.5</td>
<td>15</td>
<td>kW</td>
</tr>
</tbody>
</table>

Figure 7 shows the optimized results for the two types of coupled systems. We see that by using a lifting system, the optimal fuel flow at the supersonic flight condition is improved between 1.5% to 2.6%. The only ways for the non-lifting system to meet the heat transfer requirement is by changing the bypass airflow or duct size, whereas the lifting system can also change the quality and quantity of the thermal load.

Tables 2 and 3 show the optimal design variable values. Examining the optimal designs, as the heat transfer requirement increases for the non-lifting system, the optimizer chooses to decrease FPR that lowers the temperature of the air in the bypass duct, allowing for more heat transfer with the heat exchanger. On the other hand, the lifting system can have a higher FPR while still meeting the designated heat transfer, leading to a more efficient engine. The addition of the lifting system allows the optimizer to find the thermodynamically optimal engine design for each heat transfer requirement. This is partially possible because we are only considering engine performance at a single flight condition.

Figure 7. The lifting system allows the optimizer to find a lower fuel burn across all heat transfer requirements
Table 2. Optimal Design Variables Values for the Non-Lifting System

<table>
<thead>
<tr>
<th>Heat transfer, kW</th>
<th>Design thrust, lbf</th>
<th>OPR</th>
<th>FPR</th>
<th>Main BPR</th>
<th>Fuel flow, lbm/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>31.584</td>
<td>35.0</td>
<td>3.050</td>
<td>0.598</td>
<td>2.280</td>
</tr>
<tr>
<td>5.0</td>
<td>31.548</td>
<td>35.0</td>
<td>2.787</td>
<td>0.589</td>
<td>2.286</td>
</tr>
<tr>
<td>7.5</td>
<td>31.504</td>
<td>35.0</td>
<td>2.540</td>
<td>0.581</td>
<td>2.290</td>
</tr>
<tr>
<td>10.0</td>
<td>31.464</td>
<td>35.0</td>
<td>2.309</td>
<td>0.571</td>
<td>2.295</td>
</tr>
<tr>
<td>12.5</td>
<td>31.448</td>
<td>35.0</td>
<td>2.095</td>
<td>0.555</td>
<td>2.301</td>
</tr>
<tr>
<td>15.0</td>
<td>31.436</td>
<td>35.0</td>
<td>1.895</td>
<td>0.533</td>
<td>2.307</td>
</tr>
</tbody>
</table>

Table 3. Optimal Design Variable Values for the Lifting System

<table>
<thead>
<tr>
<th>Heat transfer, kW</th>
<th>Design thrust, lbf</th>
<th>OPR</th>
<th>FPR</th>
<th>Main BPR</th>
<th>Fuel flow, lbm/s</th>
<th>Lifting work, kW</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>31.651</td>
<td>35.0</td>
<td>5.000</td>
<td>0.643</td>
<td>2.247</td>
<td>0.62</td>
</tr>
<tr>
<td>5.0</td>
<td>31.651</td>
<td>35.0</td>
<td>5.000</td>
<td>0.643</td>
<td>2.247</td>
<td>1.94</td>
</tr>
<tr>
<td>7.5</td>
<td>31.651</td>
<td>35.0</td>
<td>5.000</td>
<td>0.643</td>
<td>2.247</td>
<td>4.24</td>
</tr>
<tr>
<td>10.0</td>
<td>31.653</td>
<td>35.0</td>
<td>5.000</td>
<td>0.644</td>
<td>2.246</td>
<td>8.00</td>
</tr>
<tr>
<td>12.5</td>
<td>31.660</td>
<td>35.0</td>
<td>5.000</td>
<td>0.644</td>
<td>2.246</td>
<td>14.18</td>
</tr>
<tr>
<td>15.0</td>
<td>31.669</td>
<td>35.0</td>
<td>5.000</td>
<td>0.644</td>
<td>2.246</td>
<td>24.67</td>
</tr>
</tbody>
</table>

The lifting system introduces additional thermal energy into the bypass duct, which causes the bypass nozzle to create more thrust than if the airflow was not heated. This slightly positive net thrust effect, coupled with the larger temperature difference possible through the lifting system, counteracts some of the negative effects associated with needing to dissipate more thermal energy. The optimal lifting work values go from 0.62 kW to 24.67 kW across this range of heat transfer requirements. Since we require more heat to be dissipated, much more lifting work is required at each optimal point.

The coupled model does not include many of the realistic considerations that are necessary when measuring total system performance. In particular, we do not account for the weight of the heat exchangers or thermal system components, which is not negligible. Additionally, the actual performance of the ACM may differ than what is modeled here, because we are using an analytic Brayton cycle with assumed efficiencies. We are considering steady-state thermal effects, though heat to flow is an inherently transient problem. We would have to integrate this engine-thermal system across multiple time points to accurately evaluate the transient performance.

5. Conclusion

In this work, we constructed a coupled engine-thermal system that models a 3-stream split-flow supersonic engine with a heat exchanger in the secondary bypass duct and an optional heat lifting system. The multidisciplinary model is developed using OpenMDAO and integrates other disciplinary analysis modules. We constructed this model in a modular way to more easily integrating it into a larger vehicle or mission-level optimization problem.

We first performed analysis sweeps to examine how the model behaves and to ensure that the trends matched physical reasoning. We then performed a series of optimizations across a range of heat transfer requirements. We compared the performance of the engine with and without the thermal lifting system and found that thermal lifting enabled fuel burn
reductions of 1.5% for 2.5 kW of heat rejection, to 2.6% for 15 kW of heat rejection. The results presented here are limited in scope, and this model could be used for a variety of other optimization studies.

Future work could optimize the heat exchanger design, examine multiple flight conditions, or introduce more physical trade-offs in the thermal system model that would lead to more realistic results for a given aircraft. This work shows that it is possible to do gradient-based design optimization of a coupled thermal-propulsive system to obtain optimal engine performance under thermal load.

Acknowledgments

The first and second authors are grateful for support from the National Science Foundation Graduate Research Fellowship under Grant Number DGE-1256280. This work is supported in part by the U. S. Air Force Research Laboratory (AFRL) under the Michigan-AFRL Collaborative Center in Aerospace Vehicle Design (CCAVD), with Darcy Allison providing technical guidance as the task Technical Monitor. We thank Justin Gray and Jon Seidel for their insightful discussions and research advice.

References

A NOVEL HYBRID ALGORITHM WITH APPLICATION IN THE KINEMATIC-BASED STRUCTURAL OPTIMIZATION OF ROBOTS

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Abstract

In this paper, a hybrid simulated annealing algorithm is proposed for optimization problems in engineering applications. The present algorithm introduces a study of the effect of hybridization between simulated annealing and artificial bee colony. The hybrid algorithm and its origins were examined through the kinematic equations of a six degree of the freedom robot manipulator, KUKA KR 100-2 comp model. The examinations showed that the performance of this type of hybridization for the structural optimization of the robots is better than the performance of simulated annealing and less than an artificial bee colony.

Key words: Structural Optimization, Kinematic Synthesis, Simulated Annealing, Artificial Bee Colony, Robotics

1. Introduction

Kinematic synthesis of a robot arm is performed by two levels, best type synthesis which is the topology of the arm [1] and minimum required dimensions should be calculated during the dimensional synthesis process[2]. Both of these two levels can be done in two separated operations or one single operation, [3] has solved the two levels in one single use of a genetic algorithm. [4] Have proposed an approach based on graph theory find the type synthesis while the optimum dimensions parameters are estimated by the Precision Position Method. On the dimensional level, [5] have dealt with shape optimization of the link of the robot using level set techniques. The start point for most of the works to conduct the type is the graph theory methods and essentials [6] where candidate topologies are enumerated to get the optimum type. In this work, a hybridization between simulated annealing and artificial bee colony optimization algorithms was achieved to produce more powerful optimization algorithm than simulated annealing. The new hybrid algorithm is proposed to calculate the optimum dimensions of a prescribed type synthesis. A well-known topology of robot arm was conducted in this text to find the dimensional synthesis of the robot depending on the kinematic equations, KUKA KR 100-2 comp [7] has a known type synthesis, and it was used here to evaluate the dimensional level corresponding to a prescribed task point. A comparative study also has been done among simulated annealing, artificial bee colony, and their hybridization. The test function for this comparison is the kinematic equations of the abovementioned manipulator.

2. Simulated Annealing SA

In 1983, Kirkpatrick developed a heuristic algorithm called simulated annealing [8-9]. The algorithm was inspired by the artificial phenomena of annealing metals where the metal is heated to a high temperature then cooled under control to get more reliable properties.

3. Artificial Bee Colony ABC

It is a metaheuristic algorithm that belongs to the swarm intelligence and has inspired by the foraging behavior of the honey bees[10]. The algorithm consists of three main sections, employed bees section, onlooker bees section, and scout bees section [11], and initially all bees are scouts. This algorithm has received too many contributions to improving its performance and this study tries to answer the question: what if we hybridize ABC with SA? What is the advantage and what are the disadvantages?
4. Hybrid Algorithm

The main point in this algorithm is employing the abandonment criteria of the artificial bee colony to the simulated annealing algorithm to produce a hybrid algorithm. In this new algorithm, a parameter called $K$ is defined for each solution in the population of solutions. Initially, the $K$ parameter has the value of zero, and if the solution does not improve after a specific number of iterations, the solution would be changed to a random solution within the search space. This specific times of iterations is called counter or $C$ parameter. This process is similar to force the employed bees to be scout bees when they cannot improve their positions after an interval of time. In the annealing process, forcing the undesired phase to be different random phase represents (theoretically) applying local energy to specific zones in the material.

```
Input: ProblemSize/ * Dimension of the search space * /
iterationsmax/ * Number of iterations * /
tempsmax/ * Maximal temperatures * /
K-- zero/ * Abandonment vector * /
Counter/ * Constant value for example 10 * /
Output: Sbest
1. Scurrent 1 ← CreateInitialSolution(ProblemSize);  
2. Sbest ← Scurrent;  
3. for $i = 1$ to iterationsmax do  
4.   $S_1$ ← CreateNeighborSolution(Scurrent);  
5.   tempcurrent ← CalculateTemperature($i$, tempsmax);  
6.   if Cost($S_1$) $\leq$ Cost(Scurrent) then Scurrent ← $S_1$;  
7.   if Cost($S_1$) $\leq$ Cost(Sbest) then Sbest ← $S_1$;  
8.   end  
9.   else if Exp( (Cost(Scurrent) − Cost($S_1$))/ tempcurrent ) $> \text{Rand}()$ then Scurrent ← $S_1$;  
10.  else $K$ ← $K$ + 1  
11.  end  
12.  if $K$ $>$ Counter then $N_i$ ← CreateNeighborSolution, $K$ ← zero  
13.  if Cost($N_i$) $\leq$ Cost(Sbest) then Sbest ← $N_i$;  
14.  end  
15. return Sbest;
```

Figure 1. The pseudo code of the proposed hybridization

5. Structural Optimization of 6 DOF Type

In this section, we will start with a simple example of structural optimization to understand the basic idea before moving to the next section where there is a more complex problem. The type synthesis of robot arm KUKA KR 100-2 comp. Figure 2. is introduced to the dimensional synthesis process. The frames assignment and robot specifications are shown in Figure 3 and Table 1 respectively. The problem can be mentioned as follows: Consider
Figure 4. For a given task point in the Cartesian space \([x=1560 \, \, y=0 \, \, z=1005]\), the configuration is the same as in Figure 3 \([\theta_1 = 0 \, \, \theta_2 = -90 \, \, \theta_3 = 90 \, \, \theta_4 = 0 \, \, \theta_5 = 0 \, \, \theta_6 = 0]\) and assume we do not know how much the lengths of the links of the arm, the question is what is the optimum length of links \(A, B, C,\) and \(D?\) For this problem, it is usual to substitute Denavit parameters from Table 1 into the Denavit matrix \([12]\) to get the homogeneous transformation matrices for each link. Equations (1) to (6) be the homogenous transformation matrices of the six links while Multiplication all the six abovementioned matrices can estimate the homogeneous transformation matrix which relates the end-effector to the base frame \((T)\) in equation (7).

\[
H_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(1)

\[
H_2 = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(2)

\[
H_3 = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

(3)

\[
H_4 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(4)

\[
H_5 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(5)

\[
H_6 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(6)

\[
T = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33} \\
0 & 0 & 1
\end{bmatrix}
\]

(7)

where \(C\) and \(S\) denote \(\cos()\) and \(\sin()\) functions respectively while \(r^a\) elements are the components of the orientation matrix which represent the orientation of the end-effector with respect to the base frame and \(x, y,\) and \(z\) coordinates are indexed as follows

\[
x = T \, (1,1)
\]

(8)

\[
y = T \, (2,1)
\]

(9)

\[
z = T \, (3,1)
\]

(10)

according to (8), (9), and (10) the objective function \(f\) can be formulated as

\[
f = \sqrt{(x - 1560)^2 + (y - 0)^2 + (z - 1005)^2}
\]

(11)

The objective function is nothing but the direct distance between the end-effector coordinate and the task coordinates, and equation (11) represents the distance law between two points in the space.

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For this problem, the number of variables is four; an initial guess is that the search space \( S \) can be \( 200 \leq S \leq 1800 \), and already the objective function is modelled from equation (1) to (11). Thus, problem parameters are known.
for optimization algorithms. Table 2 and Table 3 show the parameters values of simulated annealing and artificial bee colony algorithms. The results of the optimum lengths of the links are shown in Table 4 where they represent the optimum structure of the KUKA robot for the given task point. By drawing a robot configuration similar to that one in Figure 4 using lengths of links which are got from optimization algorithms, we can evaluate the performance of each algorithm. The optimum length of link B is 1050 mm by all the optimization algorithms but for the rest of the other link there are different sets of lengths, even the same algorithm may return different sets for each trail. Though we have a different set of lengths A, C, and D the summation of them in all cases is 1560 mm which is the x-axis of the task point, and that what it is desired. The total length of the links is optimized to enable the arm to reach the goal, and that means a robot with minimum mass, minimum cost. The search space was $200 \leq S \leq 1800$ for all the four-link, but it is possible to define a search space for each variable representing the length of a link.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>Iterations</td>
<td>20000</td>
</tr>
<tr>
<td>Maximum Temperature</td>
<td>2000</td>
</tr>
<tr>
<td>Cooling Rate</td>
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</tr>
<tr>
<td>Number of Population</td>
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</tr>
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<table>
<thead>
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</thead>
<tbody>
<tr>
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<td>1000</td>
</tr>
<tr>
<td>Number of Population</td>
<td>300</td>
</tr>
<tr>
<td>Onlooker Bees</td>
<td>150</td>
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<tr>
<td>Employed Bees</td>
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<tr>
<td>Limit</td>
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<table>
<thead>
<tr>
<th>Link</th>
<th>Simulated annealing</th>
<th>Artificial bee colony</th>
<th>Hybrid algorithm</th>
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<tbody>
<tr>
<td>A mm</td>
<td>363.44</td>
<td>200.05</td>
<td>379.29</td>
</tr>
<tr>
<td>B mm</td>
<td>1049</td>
<td>1050</td>
<td>1053.1</td>
</tr>
<tr>
<td>C mm</td>
<td>577.57</td>
<td>200</td>
<td>263.78</td>
</tr>
<tr>
<td>D mm</td>
<td>606.01</td>
<td>1159.9</td>
<td>924.77</td>
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<tr>
<td>Fitness</td>
<td>13.0026</td>
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<td>8.4884</td>
</tr>
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</table>

6. Conclusions

By following rules of the structural optimization and dimensional synthesis, the cost of the components can be reduced to the minimum value, and this is important for economic and environmental reasons. Kinematic equations of the robot manipulators are one of the quantities that can be optimized to reach the optimum structure for the robot arm. While there are many optimization algorithms, we proposed here a hybridization between two known algorithms. The
new hybrid algorithm which is presented in this work is a combination of the simulated annealing and artificial bee colony optimization algorithms. A comparative study has been done among the new hybrid algorithm and its origins. As we expect, the hybrid algorithm is better than simulated annealing but less powerful than the artificial bee colony, and all of them are less powerful than the harmony search algorithm.

Acknowledgments

The described article was carried out as part of the EFOP-3.6.1-16-2016-00011 “Younger and Renewing University-Innovative Knowledge City-institutional development of the University of Miskolc aiming at intelligent specialization” project implemented in the framework of the Széchenyi 2020 program. The realization of this project is supported by the European Union, co-financed by the European Social Fund.

References

A COMBINED LEVEL SET-XFEM AND DENSITY-BASED TOPOLOGY OPTIMIZATION APPROACH

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Abstract

Topology Optimization (TO) using Level Sets (LS) in combination with an immersed boundary technique provides a crisp definition of material interfaces. In addition, better geometry resolutions than density-based methods can be achieved in final designs using similar mesh sizes. However, these methods lack the ability to control the local feature size, and final designs may strongly depend on the initial topology. On the other hand, unlike LS-based TO, in density-based TO no initial hole seeding is required, and feature size control can be achieved conveniently through appropriate filtering and projection. In this manuscript, we introduce a formulation that couples two independent design variable fields corresponding to the LS and density-based approaches via a penalization term in the objective. In this combined approach, the design is advanced by both shape sensitivities and sensitivities with respect to material parameters. The former sensitivities are localized in the vicinity of the interface, while the latter exist in the entire design. A numerical example shows that the proposed scheme is capable of hole nucleation and minimum feature size control.

Key words: Explicit Level-Set, XFEM, Combined Level Set Density, Hole nucleation, Minimum Feature Size Control

1. Introduction

Topology optimization (TO) allows to systematically optimize the conceptual layout of a body based on a basic description of a design problem. Density-based TO, and the Solid Isotropic Material with Penalization (SIMP, [9]) approach in particular, has become one of the most popular forms of TO. Yet, due to its nature, material interfaces are not well-defined. Smeared or jagged boundaries as a product of density-based mappings of material properties on fixed background meshes introduce inaccuracies when modeling physical responses. Moreover, additional post-processing is needed to obtain the final design. On the other hand, level set LS-based TO combined with a body-fitted mesh or an immersed boundary technique provide a crisp (thus, unambiguous) interface definition. Shape sensitivities are computed on the material interfaces (unlike density-based, in which design sensitivities exist globally) to advance the optimization problem. Nevertheless, final designs are initial guess dependent, and enforcing feature size control is not trivial.

While comparisons of density-based and LS-based TO methods have been presented by [7], a combination of them has been largely unexplored. The point of departure for combining these two methods is defining two independent sets of design variable fields for the LS and the density distribution. For example, as demonstrated in [2], a LSF can be used to distinguish between phases I (solid) and II (void), while a classical density interpolation scheme is introduced to differentiate materials within the solid phase. In [5], a classical SIMP approach is used for material interpolation while a LS method is employed to exclude void domains from the TO process. Similarly, in [1] non-design spaces described by CutFEM are embedded into the design space optimized by density-based TO. Recently, a combined approach that achieves minimum length scale by adopting geometric constraints was presented in [16].

In this paper, we investigate a combined LS-density TO approach where both the LS and density fields are being optimized simultaneously to take advantage of the benefits exhibited by either method separately. In contrast to the combined LS-XFEM-SIMP approach discussed above, a loosely-coupled relationship between the two independent design variable fields introduced as a penalization term in the objective is studied.

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The remainder of the paper is organized as follows: section 2 provides the required definitions of the design variable fields used. Section 3 details the coupling scheme adopted. Section 4 presents the discretization/analysis method employed. A numerical example is presented and discussed in section 5, and section 6 summarizes the paper.

2. Combined LS-XFEM-SIMP Topology Optimization

The combined LS-XFEM-SIMP approach developed includes two independent sets of design variables, \( s^d \) and \( s^r \), for the LS-XFEM and SIMP approaches, respectively. The vector of design variables is defined as \( s = [s^d, s^r] \).

2.1 LS-XFEM Design Variables

The material layout composed of two phases in a design domain, \( \Omega_0 \), is given by

\[
\phi(X) \begin{cases} < 0, & \forall X \in \Omega_0 \\ \geq 0, & \forall X \in \Omega_0^\circ \\ = 0, & \forall X \in \Gamma_{\text{int}} 
\end{cases}
\]

(1)

Where \( \Omega_0 \) and \( \Omega_0^\circ \) are the material domains of phases I and II, respectively, such that \( \Omega_0 = \Omega_0^\circ \cup \Omega_0^\circ \). The interface, denoted by \( \Gamma_{\text{int}} \), corresponds to the zero LS iso-contour \( \phi(X) = 0 \), and is obtained from a parametrized LS function.

The nodal LS design variables, \( s^d \), are updated via a mathematical programming technique. A linear filter (\([10]\)) is used to define an explicit relationship between the LS design variables and the LSF,

\[
\phi_i = \sum_{j=1}^{N} \omega_j s^d_j, \quad \omega_j = \max(0, r_{ij} - |X_i - X_j|)
\]

(2)

where \( N \) is the number of nodes within the filter radius, \( r_{ij} \), and \( |X_i - X_j| \) is the Euclidean distance between nodes \( i \) and \( j \). Furthermore, to improve stability in the optimization process, the LS field is regularized via a signed distance function obtained by the heat method as detailed in [4].

2.2 SIMP Design Variables

A second nodal design variable field composed of fictitious densities, \( 0 < s^r < 1 \), is introduced for interpolation of the physical material properties. To increase numerical stability and enhance convergence of the optimization problem, a linear filter similar to the one presented in Eq. (2), is introduced to compute filtered densities, \( \tilde{\rho}_i \), from the density design variables, \( s^r \). In addition, to counteract the blurriness in the physical material properties introduced through linear filtering, the smoothed Heaviside projection scheme proposed in [11] is applied;

\[
\tilde{\rho}_i = \frac{\tanh(\eta_r (\rho - \eta_r)) + \tanh(\eta_r (1 - \rho))}{\tanh(\eta_r (1 - \rho)) + \tanh(\eta_r (\rho - \eta_r))}
\]

(3)

Using filtering of the density design variables together with the appropriate projection threshold parameter, \( \eta_r = 0 \), provides feature size control. A continuation approach which gradually increases the projection sharpness, \( \eta_r \), is employed.

For a two-material problem, the SIMP material interpolation (as described by [9]) is stated as;

\[
\kappa(\tilde{\rho}) = \kappa^A + (\kappa^B - \kappa^A) (\tilde{\rho})^{\beta_r}
\]

(4)

The nonlinear power law in Eq. (4) defines a generic material property, \( \kappa \), as function of the properties corresponding to material A and B, denoted by \( \kappa^A \) and \( \kappa^B \), respectively. In this work, we focus on material-void problems; hence, material A represents any solid material property and material B, void. The SIMP exponent is denoted by \( \beta_r \).

2.3 Combined LS-XFEM-SIMP Optimization Problem

The minimization problem function of the design variables, \( s \), and the state variables, \( u \), is formulated as follows;

\[
\min_z(s) = \omega_1 M(s) + \omega_0 \int_{\Omega_0} PR_n(s) dV + \omega_1 \int_{\Omega_0} \tilde{\rho}_v(s) dV
\]

\[
s, \quad g(s, u) = \frac{u(s, u)}{u} - 1.0, \quad s = [s^d, s^r], \quad u \in \mathbb{R}^N
\]

\[
s^d \in \Pi = \{ R^{s^d} | s^d \leq s^d \leq s^d \}, \quad s^r \in \Pi = \{ R^{s^r} | s^r \leq s^r \leq s^r \}
\]

(5)

The objective is composed of four weighted components such that \( \omega_1 + \omega_0 + \omega_1 + \omega_1 = 1.0 \). A globally enforced mass measure, \( M = M^d + M^v \), where \( M^d \) and \( M^v \) represent the total mass in phases 1 and 2, respectively, is applied. This
mass measure allows us to combine the hole nucleation process (i.e., generating void regions in an initially material-only domain) with minimum feature size control through filtering and projection of the density design variables. The remaining components represent the perimeter control penalty to avoid the emergence of irregular geometric features; the LS regularization penalty \( p_H \) integrated over the design domain for a smooth, regularized LS field (as detailed in [4]), and the coupling penalty \( \bar{p}_H \) also integrated over the design domain. The formulation for the coupling term is detailed in the next section. A constraint that imposes an upper bound on the strain energy, \( u_* \), is introduced, where \( u_* \) is maximum feasible strain energy. Both LS-XFEM and SIMP design variables are limited by their corresponding lower and upper bounds, \( [\bar{\epsilon}_l, \bar{\epsilon}_u] \) and \( [\bar{\epsilon}_l, \bar{\epsilon}_u] \), respectively. The number of LS-XFEM and SIMP design variables, as well as the number of state variables, are noted by \( N', N'_*, \) and \( N_0 \), respectively. Note that the structural density contributing towards the mass of the structure is considered globally in phases I and II (entire domain), while the displacements, and therefore the strain energy, is only computed in the solid domain (phase I).

3. Coupling Strategy

A combined approach that loosely couples LS and density design variables, as proposed in [8], is discussed hereinafter.

3.1 Coupling Penalty Formulation

The penalty \( \bar{p}_H \) introduced in Eq. (6) is formulated such that it couples the filtered and projected nodal densities, \( \hat{\phi} \), with the filtered nodal LS values, \( \phi \), as follows:

\[
\bar{p}_H = \max\left((-\phi + \bar{\phi}_n), 0\right) \cdot \max\left((-\hat{\phi} + \bar{\hat{\phi}}_n), 0\right)
\]  

(6)

Note that nodal filtered LS values are chosen to operate on a smoothed field, while nodally filtered and projected densities are the preferred choice to improve smoothness and gain feature size control. The LS-XFEM and SIMP thresholds are denoted by \( \bar{\phi}_n \) and \( \bar{\hat{\phi}}_n \), respectively. The penalty function of Eq. (6) linearly penalizes any negative LS values below \( \bar{\phi}_n \) when low density values below \( \bar{\hat{\phi}}_n \) are present. The proposed penalty exhibits gradients with respect to both \( \hat{\phi} \) and \( \phi \); which is necessary to provide non-zero design sensitivities that lead to the desired hole nucleation. With this penalty, we achieve positive LS values (i.e., void), and thus hole nucleation, in areas of low density (i.e., weak material). Similar to a penalty on the lower end of the density range, an additional “upper” penalty was investigated. This penalty term enforces low densities in void regions (positive LS values) such that the level set field aligns with volumes of high density material and preserves the feature size control achieved by the density projection scheme. However, the upper penalty is not required if the mass constraint includes both phases. For this reason, only one penalization term is introduced. More details regarding the effects of an upper penalty can be found in [8].

3.2 Smoothed Coupling

To obtain a differentiable penalty formulation, the penalty in Eq. (6) is approximated to a smooth function by using a Kreisselmeier-Steinhauser (KS) function ([12]) as follows:

\[
\tilde{\bar{p}}_H = \frac{\lambda}{\bar{\lambda}_K} \ln \left[ \exp \left( \frac{\bar{\lambda}_K}{\bar{\lambda}_K} \right) + 1 \right] \ln \left[ \exp \left( \frac{\bar{\lambda}_K}{\bar{\lambda}_K} \right) + 1 \right]
\]  

(7)

In Eq. (7), the KS parameter is typically set to a sufficiently large value such that the smooth approximation closely resembles Eq. (6), but not too large to avoid numerical instabilities associated with extremely large numbers result of the exponential function evaluation.

3.3 Influence of Density Projection

When using the smoothed Heaviside projection, small density values are projected to zero. This generates a positive LS field through the effect of the penalty of Eq. (7). Thus, the density projection parameter \( \gamma_H \), which controls the effect of the projection, has a strong influence on the convergence of the proposed combined approach. An appropriate projection can drastically accelerate the hole nucleation process by removing small oscillations in the density values.

4. Discretization Technique

The XFEM ([18]), a FE-based immersed boundary technique, is used for discretization of the physics on a non-
conforming background mesh. A generalized Heaviside enrichment strategy where the degrees of freedom within each unique subphase are approximated using standard FE shape functions are employed. Multiple levels of enrichment are used to avoid spurious coupling or load transfer between disconnected material subdomains. Using this approach, the state variable vector \( \mathbf{u}_i(X) \) at node \( i \) of a two-material problem (phase I, phase II) is approximated as:

\[
\mathbf{u}_i(X) = \sum_{l=1}^{L_i} \left( H(-\phi(X)) \sum_{k=1}^{N_k} N_k(X) \delta_k \mathbf{u}_i^{k,l} + H(\phi(X)) \sum_{k=1}^{N_k} N_k(X) \delta_k \mathbf{u}_i^{k,l} \right)
\]

(8)

where \( H \) is the Heaviside function as a function of the LS value \( \phi(X) \) defined as:

\[
H(\phi) = \begin{cases} 
1, & \forall \phi(X) > 0 \\
0, & \forall \phi(X) < 0 
\end{cases}
\]

(9)

The maximum number of enrichment levels is denoted by \( L_i \); \( N_k(X) \) is the elemental shape function; and \( \delta_k \) is the Kronecker delta which selects the active enrichment level \( q \) for node \( k \). \( \delta_k \) ensures that displacements at node \( k \) are only interpolated by a single set of degrees of freedom (DOFs) defined at node position \( X \) such that the partition of unity principle is satisfied. The number of nodes per element is denoted by \( N_v \).

Boundary and interface conditions are applied weakly in this study using the unsymmetrical version of Nitsche’s method ([13]). Weakly enforced boundary and interface conditions are essential in LS-XFEM TO where the material phase of a domain boundary at which Dirichlet boundary conditions are applied may change. Furthermore, face-oriented ghost penalization ([14]) is used to stabilize the XFEM discretization in the vicinity of the interface. Numerical instabilities arise in the XFEM when the material interface moves too close to a FE node, leading to a vanishing zone of influence of certain degrees of freedom. Face-oriented ghost stabilization cures this ill-conditioning independent of the intersection configuration. For more details about the generalized Heaviside enrichment strategy employed in this study, the interested reader is referred to [2-3, 6].

5. Numerical Example

The proposed approach is investigated in this section using a single material, solid-void example problem where a SIMP scheme is combined with explicit LS-XFEM TO for structural TO. Initially, uniform LS and density fields of \( \sigma = -1.0 \) and \( \sigma = 1.0 \) are used to have a material-only domain and satisfy the strain energy constraint. For this reason, only (global) design sensitivities with respect to the density design variables exist until holes are nucleated. The LS design sensitivities are localized and only non-zero in the vicinity of the XFEM interface once the hole nucleation process has initiated. The coupling penalty function is activated after the density field has sufficiently converged, close to a 0-1 design.

5. 1 Beam Problem

The classical beam benchmark problem in structural TO ([17]) is used to study the formulation of Eq. (5) in 2D. The design domain is of size \( 240 \times 40 \), with a traction load of \( -10.0 \) applied at the top center of the domain over a length of \( 3.0 \). Units are self-consistent and, therefore, omitted. The structure is simply supported on both ends on the bottom of the domain. Due to the symmetry of the design problem, only half of the domain is modeled and optimized. Symmetry boundary conditions are applied weakly along the y-axis at the center of the domain. The structural response is described by linear elasticity (plane stress) using a Young’s modulus of 2000.0 and Poisson’s ratio of 0.4. An unstructured mesh of element size \( h = 1.0 \) was employed with bi-linear elements. The reference strain energy chosen for this problem is \( \bar{\varepsilon} = 20.0 \). The total number of design iterations is 600 to let the problem converge to a relative difference below \( 10^{-4} \). The SIMP exponent \( \beta = [2.0, 3.0, 5.0, 7.0, 9.0, 11.0, 13.0, 15.0] \), and the projection parameter \( \gamma_p = [0.001, 0.01, 0.05, 0.07, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0] \), are increased through 6 continuation steps of 90 design each of the iterations. A KS parameter of \( \beta_k = 75.0 \) and threshold values of \( \phi_k = \rho_k = 0.1 \) are used. The weights of the individual objective contributions are \( w_i = [0.93, 0.01, 0.1, 0.05] \) after the second continuation step since the weight of the coupling penalty is set to zero in the first continuation step. The optimization problem is solved using a gradient-based nonlinear programming scheme, namely the Globally Convergent Method of Moving Asymptotes (GCMMA, [15]), with no inner iterations.
Figure 1. Solutions of the 2D beam problem \[^{[8]}\] (reflected for illustration purposes). (a) Shows the initial design of a pure LS-XFEM approach, and (b) is the corresponding final design. (c), (e), (g), (i) Show the final density field obtained by the combined approach with a filter radius of 1.8 h, 2.4 h, 3.2 h, and 6.4 h, respectively. (d), (f), (h), (j) the corresponding solid-void domain decompositions for filter radius of 1.8 h, 2.4 h, 3.2 h, and 6.4 h, respectively. See reference \[^{[8]}\] for a more detailed description.

Figures 1 (a) and (b) show the initial and optimized design for a pure LS-XFEM configuration. This approach requires a significant number of holes to be seeded at the beginning of the optimization problem. Furthermore, no explicit control of feature size is observed (see Figure 1 (b)). On the other hand, the resulting designs using the combined LS-XFEM-SIMP approach (Figures 1 (c)-(j)) show that increasing the filter radius on both the density and LS design variables removes small structural members from the final design and regularizes the overall final geometry. Figure 1 (d), (f), (h) and (j) show the final designs; while Figures 1 (c), (e), (g) and (i) display the corresponding density fields. No initial seeding is required since the density field provides appropriate guesses for hole nucleation. Moreover, it can be seen that the proposed coupling penalization instructs the XFEM interface to trace the density field without undercutting it, i.e., placing high-density material within the void region. Undercutting is avoided by the global mass constraint. Densities in the void regions contribute only towards the mass measure, and not towards the stiffness of the structure.

Note that the attained minimum features (shown in Figure 1 in red circles) differ from the applied filter radii. The observed structural members of reduced thickness are a consequence having only selected nodal density design variables active (before filtering and a projection) at the final design. The proposed combined approach inherits this limitation of density-based TO. A robust optimization approach can be employed to achieve exact minimum feature size controlled by the linear filter radius on the density design variables. Also, the transition region between high-density and low-density material is inherited from the density-based approach and can only partially be mitigated by the XFEM where subphase constant material properties are used. For this reason, a small fraction of elements with intermediate densities remains in the vicinity of the interface at the converged stage. A more detailed discussion of the presented results can be found in \[^{[8]}\].

6. Summary

A combined LS-XFEM and SIMP TO approach that provides a crisp material interface and requires no initial seeding while achieving a minimum feature size through linear filtering and projection of the density field was proposed. Downsides of either method (e.g., fuzzy interface, need for initial hole seeding) are eliminated by coupling the LS and density design variable fields through a penalty term in the objective function.
The combined approach only introduces void inclusions where needed and therefore reduces the number of intersected XFEM elements to the required minimum. This eliminates the drawback of pure LS-XFEM TO where a large number of initial holes as a starting point is typically used, which significantly increases the computational cost due to a large number of intersected elements (17). Furthermore, additional computational cost of having twice as many design variables is partially offset by the fact that no void inclusions need to be seeded at the beginning.

Overall, the combined LS-density method showed promising performance in terms of computational efficiency, robustness and analysis accuracy. As shown in [8], the proposed combined approach can also be applied in 3D problems. Without the loss of generality, this approach can be extended to incorporate both, multiple LSs and/or multiple density interpolation schemes.

Acknowledgments

All authors acknowledge support of the Air Force Office of Scientific Research (grant FA9550-16-1-0169). The first author acknowledges support from NSF (Grant 1463287). The third author acknowledges support from the Defense Advanced Research Projects Agency (DARPA) under the TRADES program (Agreement HR0011-17-2-0022). The opinions and conclusions presented in this paper are those of the authors and do not necessarily reflect the views of the sponsoring organizations.

References

CO-EVOLUTIONARY DESIGN PROCESSES APPLIED TO
BUILDING SPATIAL DESIGN OPTIMIZATION

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Abstract

Building spatial design is in practice a co-evolutionary design process. To optimize a building spatial design, an
evolutionary algorithm can be used, but the search space is large and complex. In simulations of co-evolutionary
design processes, the size and complexity are not hindering the search. Such simulations are in the presented work
proposed for finding search spaces that are small but still contain high quality solutions.

Key words: Building Design, Co-evolutionary Process, Evolutionary Algorithm, Structural Design, Thermal Design

1. Introduction

Building Spatial Design (BSD) entails the search for the spatial layout and the dimensions of a building for a given set
of design requirements. In design practice, it is traditionally performed as a co-evolutionary design process. Here, co-
evolution is the principle in which not only the design solution can change, but also the design requirements, and
typically the solution and its requirements interact [1]. For example, for a building spatial design a structural design is
generated such that a structural performance can be computed. If, in the structural model of this example, components are modified, removed, or added in order to improve the structural performance (e.g. by topology optimization), it is possible that a space in the building spatial design cannot exist or must be modified before it can accommodate the changes made to the structural design. Changing the BSD, consequently changes the design
requirements with which the structural design was generated. Building spatial design can also be formulated as an
optimization problem, although the search space that contains all possible solutions is large and complex. Modern
optimization techniques can search for building spatial designs only in a search space which size has been reduced,
which is typically achieved with so-called super structure. A super structure is a representation of the design problem in which design variables are fixed or limited to a certain range [2]. Unfortunately, a super structure may
exclude Pareto optimal points. In earlier work a super structure free optimization technique named simulation of co-
evolutionary design processes (SCDP) together with a super structured evolutionary algorithm (EA) has been applied
to building spatial design optimization [3]. The work suggested that SCDP may be able to find better solutions than
the EA (this was also observed in [4]), albeit that they lie outside of the super structure of the EA. It was also observed
that in the same search space the EA is better than SCDP in finding high quality solutions.

In this work, the ability of SCDP to improve a design is investigated. Additionally, it is studied whether the search
direction of SCDP can be controlled. If so, it would be possible to find a super structure for an EA that is suitable for
specific objectives. The studies performed in this work will be in the scope of structural building design and thermal
building design. Therefore, two objectives are defined: minimizing the strain energy; and, minimizing the sum of the
heating and cooling loads in the building. A parameter study is employed for the investigation into SCDP, which is
introduced in section 2, together with the problem representations, objective evaluations, and the SCDP and EA
algorithms. The results are then presented in section 3, and finally, in section 4 the conclusion and outlook are given.

2. Methodology

The methodology of the presented work is presented as follows. First, the used problem representations for a
building spatial design are given. Thereafter, the evaluation of the optimization objectives is explained. Accordingly,
two optimization methods are introduced, an evolutionary algorithm and SCDP. Finally, the setup of a parameter
study is presented which will give insights into the ability of SCDP to improve designs and steer the search.

2.1 Building Spatial Design Representations

Two building design representations have been developed within the research framework of which this work is a part. Both representations are limited in the sense that they can only represent cuboid spaces that are arranged in an orthogonal grid. This limitation is adopted to simplify various optimization phenomena like mutation/modification and constraints. As such, research (like the work presented here) need not be distracted by special cases introduced by e.g. round spaces. The first representation to mention is the so-called “SuperCube” (SC) representation, which is a collection of cells that is defined by a 2D grid. The grid can be dimensioned for each column/row in each direction individually. Each space is then defined by a different bitmask, with which each cell is activated or de-activated for that space. This representation has a fixed number of variables, making it well suited to traditional EA’s. The second representation is the so-called “Movable and Sizable” (MS) representation, which is a collection of spaces. Each space in the MS representation is defined by two vectors, location and (orthogonal) dimensions. A spatial design in MS representation can—in contrast to the SC representation—be easily imagined by the human mind. This is useful for SCDP, in which modification steps are defined using the intuition and experience of an engineer. For the interested reader, a detailed description and visuals of both representations can be found in [5].

2.2 Optimization Objectives

Two optimization objectives are defined, minimal strain energy for Structural Design (SD) and the minimal sum of heating and cooling loads for building physics (BP). These objectives cannot directly be obtained from a BSD itself. Therefore, so-called design grammars are used to generate the structural and the building physics model, these are detailed below.

The structural design grammar places a flat shell component that represents a concrete structure (\(E = 3,000\)\(\text{e1}\)\(\text{N/m}^2\); \(t = 150\) mm; \(v = 0.3\)) at the location of each wall and floor in the building spatial designs. Wind loads (pressure: 1.0 kN/m\(^2\); suction 0.8 kN/m\(^2\); and shear 0.4 kN/m\(^3\)) are placed on the external flat shells. Four wind load cases are defined, one for each azimuthal direction (+\(x\), +\(y\), −\(x\), −\(y\)). Moreover, a life load (5.0 kN/m\(^2\) in −\(z\)-direction) load case is added to the model on each floor in the structural model. Constraints in \(x\)−, \(y\)−, and \(z\)−direction are placed on horizontally oriented edges that have a \(z\) coordinate at or below zero (\(z \leq 0\)). Each flat shell component is meshed into 3 by 3 quadrilateral elements describing normal−, shear−, and bending action using \(2 \times 2\) integration points (Gaussian quadrature). After meshing, the resulting system is solved by using direct sparse LTL Cholesky factorizations. The SD objective is then calculated by taking the total sum of strain energy (Nnm) over all elements in the structural model. For a space’s performance, the strain energy of all elements that border a space is summed, note that in this way one element can be considered for one or two spaces simultaneously.

The building physics design grammar places a construction at the location of each wall and floor in the building spatial model. Each construction has a thickness of \(t = 150\) mm, a specific weight of \(\gamma = 2400\) kg/m\(^3\), a specific heat capacity of \(C = 850\) J/(K•kg), and a thermal conduction coefficient of \(\lambda = 1.8\) W/(K•m), which represents the thermal behavior of concrete. Additionally, an insulation layer is added on the exterior of the external walls and floors (\(t = 150\) mm; \(\gamma = 60\) kg/m\(^3\); \(C = 850\) J/(K•kg); \(\lambda = 0.04\) W/(K•m)). The BP model is a Resistor Capacitor (RC) network, in which, constructions and spaces are each modelled by one temperature point. The heat capacity of a construction or space is modelled by a grounded capacitor. Connections between spaces and constructions are modelled with a resistance. At each space’s temperature point, an ideal power source is modelled with a capacity of 100 W/m\(^3\) for both cooling and heating. If the temperature of a space rises above a setpoint of 25°C, cooling is activated, if it drops below a setpoint of 20°C, heating is activated. The outdoor temperature in the model is modelled with real world measurements obtained at De Bilt in The Netherlands [6], and for the ground temperature a constant temperature of 10°C is modelled. Two periods of each 3 days (72 hours) are simulated, a typical warm period (July 2nd-4th 1976) and a typical cold period (December 30th 1978-January 1st 1979). A warm up period of four days is prepended to both simulation periods in order to start the periods with appropriate initial temperatures. The RC-network is described by a system of ordinary differential equations, which is solved by the 5th order Dormand-Prince algorithm using error control. The BP objective for each space is then calculated as the sum of heating and cooling power over the simulated periods. The objective value for the entire building is calculated as the sum of the objective
value per space over all spaces. A more detailed explanation of the design grammars and the evaluation of objectives can be found in [5].

2.3 Evolutionary Algorithm

In this work the tailored SMS-EMOA algorithm published in [7] is used for multi-objective optimization. The algorithm uses the SMS-EMOA algorithm with tailored mutation an initialization operators, such that these do not generate infeasible designs. For constraints, specific parameter configurations, and other details the reader is referred to [7].

2.4 Simulation of Co-evolutionary Design Processes (SCDP)

An algorithm that is designed to simulate a co-evolutionary design processes are introduced next. In principle, the algorithm removes a number of spaces, and splits an equal amount of the remaining spaces. For this, it looks at the objective values per space and it takes two arguments; a space removal ratio, here denoted by $\eta$; and a sorting method, denoted by $\eta$.

The sorting parameter $\eta$ specifies how a certain objective is sorted, and which objectives are considered in the sorting process. An objective is included in the sorting process if it is supplied with two two-letter terms, of which the first term specifies the objective itself ("sd" or "bp") and the last the ordering. For the sorting process, the objective values are normalized between 0 and 1 in two different ways: "lh", the minimum is normalized to 0 and the maximum is normalized to 1; or vice versa "hl", maximum to 0 and minimum to 1. If both objectives are accounted for the parameter is a concatenation of two pairs of terms, where each pair is concatenated with a "+". For example $\eta = "sd\_lh+bplh"$ denotes that both SD and BP objectives are included, and their respective minima are normalized to 0 and their maxima to 1. Once the normalized values are computed, two sorted lists are computed; the "best" list contains—per space—the Euclidian distance from a vector containing the normalized objective value to the utopian point; and the "worst" list is similar, but contains distances in reference to the dystopian point. Here the utopian point is the 0 vector and the dystopian point is the $\overline{1}$ vector, where each vector’s dimension equals the number of considered objective values.

Space removal is based upon a selection from the "worst" list, where the number of removed spaces is $n_{del}$ and the removed spaces are the $n_{sel}$ spaces with the lowest values in the list. Here, $n_{del}$ is set to $n_{sel}$ if $n_{sel} / n_{tot} \geq 0.5$ and is set to $n_{sel}$ otherwise, where $n_{sel}$ is a lower bound of spaces to be removed, $n_{tot}$ an upper bound, and $n_{tot}$ the total amount of spaces in the BSD. The values for $n_{sel}$ and $n_{tot}$ are determined via a bi-sectioning algorithm that finds a value $a$ and a value $b$ within a margin of 0.01 where $a < b$. Accordingly, the bounds are determined as follows: $n_{sel}$ is the number of values in the "worst" list that is less or equal to $a$, and $n_{tot}$ is the number of values in that list that are lower or equal to $b$. In each iteration of the bi-sectioning algorithm $(a+b)/2$ is assigned to $a$ if the number of values in the "worst" list that is lower than that value is less than $n_{sel}$, otherwise it is assigned to $b$.

After $n_{del}$ spaces are removed, the remaining spaces are scaled in $x$- and $y$- direction by a factor of $\sqrt{V/V_0}$ in order to restore the BSD’s volume $V$ to the original volume $V_0$. After scaling, $n_{del}$ of the remaining spaces are split, starting at the top of the sorted “best” list. When split, a space is divided in two halves along the direction of its smallest horizontal dimension, if the two horizontal dimensions are within 1% of each other, the space is split along the $x$-direction. If splitting a space would lead to a dimension smaller than 500 mm, the space is skipped, unless there are not enough spaces left to reach the criteria of $n_{del}$ split spaces. The latter prevents spaces from becoming too narrow, and also enforces one of the constraints that is imposed on the BSD for the EA optimization (see [7]).

2.5 Parameter Study

In order to assess the ability of SCDP to improve the results of optimization with EA, first the EA is employed for an optimization task. The supercube for that task is set to contain a grid of $3 \times 3 \times 3$ cells ($x, y, z$), with which 10 spaces should be defined. A volume constraint for the total building volume is set to 357 m$^3$. The EA’s evaluation budget is set to 1e+4 of which the first 50 are initialisation and the EA is run 35 times using these settings. The results from all 35 runs are depicted in Figure 1, a grey/black dot represents the performance of one evaluated design, the gradient represents at which
iteration the performance was found. Note that the axis on which the strain energy is plotted uses a logarithmic scale. The Pareto Front Approximation (PFA) is here the set of non-dominated points. Depicted with blue triangles is the PFA over all runs after all iterations, depicted with red circles is the PFA over all runs after just 100 iterations. Note that, in order to take into account stochasticity, the median of all PFAs over all runs would be representative. However, here the best solutions overall are chosen so that SCDP can be compared with the best found designs in the chosen supercube.

![Results Evolutionary Algorithm](image)

Figure 1. Results from the evolutionary algorithm and a visualization of the selection of building spatial designs

For the parameter study, three designs are selected from the EA’s results, which are depicted on the right in Figure 1. These designs serve as initialising BSD’s onto which the SCDP algorithm will be applied. In the parameter study, the sorting parameter \( \eta \) will be varied in every unique mutation of its terms, which yields 8 configurations for this parameter. Furthermore, the space removal ratio \( \eta_s \) will be varied with the following values \( \{0.1, 0.2, 0.3, 0.4, 0.5\} \). As such, the increment of 0.1 for the \( \eta \) parameter signifies one space for a BSD with 10 spaces and thus 1, 2, 3, 4, or 5 spaces can be removed. Finally, SCDP will be applied successively as well, i.e. search depth, for this parameter study SCDP will be applied successively for four iterations. This successive application yields a tree structure in the parameter study for both \( \eta \) and \( \eta_s \), however only \( \eta \) will be varied after each successive SCDP application. Altogether, \( 3 \cdot 5 \cdot 8^4 = 61440 \) configurations can be evaluated with \( 3 \cdot 5 \cdot \sum_{i=0}^{3} 8^i = 8775 \) evaluations if each node in the search tree is evaluated once.

3. Results

The results of the parameter study are shown in Figure 2, each evaluated design’s performance is indicated with a grey dot, the PFA from the EA with a blue triangle, and the selected designs (1, 2, and 3) with a red circle. The PFA of the SCDP results is depicted with a green rhombus, a selection of these is shown on the bottom right. The SCDP results are plotted in three folds, in each plot the convex hull of a distinct collection of performances is drawn. At the top left of Figure 2, a distinction is made in the initial design that was selected from the EA results. From this plot, no clear distinction can be observed with respect to the initial design, which suggests SCDP is not significantly sensitive to the initial design. When a distinction is made in the space removal ratio \( \eta \) (top right), it seems that higher ratios lead to better designs, and a ratio of 0.1 does not seem to improve at all. Finally, a distinction in the search depth shows that just two iterations of SCDP in the parameter study barely lead to better results. However, in the third and fourth iterations improvements are found. From the plots it can be concluded that it is beneficial to have at least 3 successive SCDP steps with a relatively high space removal ratio (0.4-0.5).

When looking at the supercube of the designs in the bottom right of Figure 2, it can be concluded that by using SCDP different supercube sizes have successfully been found. The newly found supercubes are lower but wider, which allows all spaces to be placed on the ground floor. This is beneficial for the structural design objective as the wind and life loads can be transferred to the boundary conditions in a more direct path, therefore generating less strain energy. Also note that the sizes of the supercube in Figure 2 (12 and 16 cells) are smaller than the size of the supercube used
for the EA (27 cells), thus the search spaces of the found supercubes are smaller whilst they contain better solutions.

Although the performances of the designs improve after using SCDP, it should also be noted that only the structural design objective seems to benefit from applying SCDP. This can be explained from the fact that a perfect cube is in this case optimal for thermal building design because it has the smallest possible surface to volume ratio in the orthogonal case. From the EA results, it can be seen that the EA can already find such optimal designs.

![Diagram](image)

Figure 2. Results of SCDP with convex hull of performances per design (t1), space removal ratio (tr), and iteration (t1).

It may be clear that SCDP is able to improve the results found by an EA, albeit in a different search space. However, for the sake of supporting an EA by finding suitable search spaces, it is of interest to steer the performance of a design into certain directions. As stated, SCDP does not seem to be able to improve the BP objective in the parameter study, but it would be interesting to see if the improvements in the SD objective can be generalized into a set of SCDP steps that guarantee an improvement. Unfortunately such a clear conclusion cannot be drawn from the presented work. Even though some sorting configurations for \( \eta \) seem to be more likely to improve a BSD, when looking at the sorting steps made for each SCDP in Figure 3 no clear preference or pattern can be observed. Additionally, the depicted design paths also contain values for \( \eta \), that—judging from the statistics of the parameter study—seem highly unlikely to improve a design. In Figure 3 it can be observed that every design path first leads to worse designs before it leads to a better design. This makes it more difficult to generalize the configurations for the sorting parameter \( \eta \).

3.1 Discussion

The presented work has showed that SCDP is able to improve designs, and as such it can be used to find suitable search spaces for optimization with an EA. However some critical remarks should be made. First, the implemented SCDP modification is naive as it is not reasoning from a physics point of view, but merely from an objective value point of view. A better modification for SD might be—for example—to split a space below a floor with high strain and to remove a wall with low normal forces. Second, no clear answer was found on the question if SCDP can steer a BSD’s performance into a certain direction. Additional studies that aim to find such SCDP configurations should be performed to conclude on that.
4. Conclusion and Outlook

In this paper, an algorithm that simulates a co-evolutionary design process (SCDP) of building spatial designs is introduced. It is reasoned that SCDP can help optimize using evolutionary algorithms (EA) by finding promising search spaces. A parameter study is performed on the SCDP algorithm, using designs that were found by the EA. It has been shown that better designs than the EA’s results can be found using SCDP, albeit that they lie in a different search space. The latter proves that SCDP can be used to suggest suitable search spaces to EAs. However, the parameter study did not prove to be sufficient in identifying the parameters that allow to steering the performance of a building spatial design into specific directions. Therefore, future work will focus on further development and research of the SCDP algorithm in order to be capable of steering a design’s performance into specific directions.

Acknowledgments

This work is part of the TTW-Open Technology Program with project number 13596, which is (partly) financed by the Netherlands Organization for Scientific Research (NWO).

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ADAPTIVE SENSITIVITY ANALYSIS STRATEGY FOR RBDO
WITH SURROGATE MODELS

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Abstract

Design sensitivity information is essential in iterative design processes. It affects not only the efficiency of sensitivity analysis, but also the convergence rate to the optimum design. The finite difference method has been widely used for design sensitivity analysis, with moving step size being a constant or as a coefficient of design variables. In the finite difference method implementation, the approximation is inaccurate such that it sometimes leads to divergence, more iterations, or higher computational costs in the iterative design process. This paper introduces a new sensitivity analysis technique for reliability-based design optimization using surrogate models. The developed method enables automated moving step size adjustment in sensitivity analysis thus optimally balancing the efficiency and accuracy. For the design point far away from the optimum design, a relatively large moving step size is selected in the sensitivity analysis to obtain a high convergence rate and a reduced computational cost. As the design points move closer to the optimum design, a smaller moving step size is selected by the trajectory of the searching process. Our numerical experiments of the reliability analysis and design have demonstrated that the proposed approach could improve both efficiency and accuracy of reliability-based design optimization employing surrogate models.

Key words: Reliability-Based Design Optimization, Kriging, Sensitivity Analysis, Surrogate Model

1. Introduction

With the increasing complexity of engineering systems along with growing high-reliability requirements, effective reliability assessment and optimization techniques become more and more critical in system design. The traditional deterministic design optimization model [1-2] has already been successfully applied in engineering design to systematically reduce the development cost and improve reliability. However, uncertainties widely exist in engineering simulations [3] and manufacturing processes [4] which shows a necessity to consider various uncertainties and to have robust and cost-effective designs using a Reliability-Based Design Optimization (RBDO) model. Traditionally, RBDO problems can be formulated as a stochastic optimization problem under probabilistic constraints. The primary challenge of an RBDO problem is the evaluations of probabilistic constraints which then have been converted into a deterministic constrained optimization problem. The failure probability has been evaluated in the main optimization loop, and reliability sensitivity assessment is conducted using different iterative and sampling methods in the inner loop. This technique usually leads to a nested optimization problem with high computational cost, referred to be as the double loop approach [5]. The two well-known methods that use this technique are the Reliability Index Approach (RIA) [6] in which the optimization problem is solved to estimate the probability of failure, and the Performance Measure Approach (PMA) [7] where the probability estimation has been transformed into a performance measure by solving the inverse problem.

Then, the surrogate models have been commonly used to alleviate the computational burden raised in the RBDO problem [8-10]. The surrogate model serves as an effective approach to reduce the cost of function evaluations by inferring the expected behavior of the objective function, and thus alleviate the computation burden. Kriging model was introduced in solving RBDO problem, and the Latin Hypercube sampling (LHS) [11-12] has been used to construct the performance function. There are many efficient methods have been applied to solving complexed engineering optimization problems, which aim to reduce the number of function evaluations for each function evaluation could be very expensive in terms of time and cost. In a study [13], surrogate models can be combined with the PMA technique in a sequential optimization and reliability analysis (SORA) [14] structure. SORA scheme
separates the reliability sensitivity analysis from the main optimization loop and transforms the RBDO problem into the sequences of deterministic optimization and reliability analysis cycles to reduce the computational cost, whereby the PMA technique is used for reliability assessment based on the Kriging models. These techniques improve the accuracy of the failure prediction and reduce system development costs, thus leading to efficient and reliable system designs.

For RBDO, design sensitivity information is essential in the iterative design processes, as it affects not only the accuracy of reliability analysis in each design iteration, but also the convergence rate to the optimum design. This paper, therefore, introduces a new sensitivity analysis technique for RBDO employing surrogate models based on an adaptive finite difference method. The method developed in this paper combines the optimization process with the resolution of the sensitivity both efficiently and accurately. Form accuracy perspective, we need to adjust the step size of the finite difference method from the surrogate. For the design point far away from the optimum design, a relatively large moving step size in sensitivity analysis is selected to obtain a high convergence rate and a reduced computational cost. As the design points move closer to the optimum design, a smaller moving step size is selected by the trajectory of the searching process. The reliability index, the distance between design points, and trajectory of the optimization process have been used as measurements to adjust the step size for finite difference method adaptively. It enables us to determine when and where to reduce the moving step size to achieve a desired balance between the efficiency of the RBDO process and the accuracy of the reliability analysis during the intermediate design iterations. The efficacy of the developed adaptive sensitivity analysis technique for reliability-based design optimization using surrogate models is demonstrated with one numerical reliability analysis and design case study through the comparisons with several existing sensitivity analysis approaches.

2. Related Work

This section gives a brief review of the reliability-based design optimization problem. Subsection 2.1 provides a review of the reliability analysis approach for the surrogate model using the Monte Carlo Simulation (MCS) method. Subsection 2.2 and 2.3 introduce the Kriging surrogate model and RBDO problem respectively.

2.1 Reliability-based Design Optimization

Reliability-based design optimization method has been broadly used in solving engineering reliability design problems. The RBDO model [15-18] can generally be formulated as

\[
\begin{align*}
\text{find} & \quad d \\
\text{min} & \quad \text{Cost}(d) \\
\text{s. t.} & \quad P(G(x,d) \leq 0) \leq 1 - \Phi(\mu) \\
& \quad d^L \leq d \leq d^U, \quad d \in \mathbb{R}^n \quad \text{and} \quad x \in \mathbb{R}^n
\end{align*}
\]

where \( d \) is a vector of design variables, \( \text{Cost}(d) \) is the objective function, \( nc \) is the number of probabilistic constraints; \( nd \) is the design variables; \( nr \) is the random variables. Superscripts ‘L’ and ‘U’ are the lower and upper bound, respectively. The RBDO can be computationally expensive due to many function evaluations of the expensive high-fidelity models that have been used to estimate the reliability of the system in each optimization iteration. In this paper, the Kriging surrogate model has been used to replace the high-fidelity model and can be adaptively refined at each iteration. In this study, we combine the Kriging model with the RBDO method which consists of two loops. In the inner loop, the reliability sensitivity analysis based on Kriging surrogate model is conducted using different adaptive sampling methods, while the outer loop optimizes the design variables and the objective.

2.2 Reliability Analysis

Considering the performance function \( G(x) \) which characterizes the system response. A positive value means that the failure occurs, and a negative value implies that the system is safe. The boundary of the positive and negative domain is called the limit state. The probability of failure can be defined as

\[
P_f = P(G(x) > 0) = \int_{G(x) > 0} f_s(x) \, dx
\]

where \( G(x) \) is the constraint functions, and \( f_s(x) \) is the joint probability density function of the system random inputs, and \( G(x) > 0 \) is defined as the failure region. The Monte Carlo Simulation (MCS) [19] provides a convenient
approximation for both reliability analysis and inverse reliability analysis because it directly approximates the \( b \) and \( G \) relationship. Here, MCS has been used to calculate the probability of failure for a given design point. We use the probability of points located in the safe region to represent the reliability of the whole system. More points located in the safe area means more reliable of the system is. Indicator function has often been used to calculate the number of points located in the safe (or failure) region. Then the system responses \( G(x) \) will be evaluated to estimate the \( P_f \) using indicator function.

\[
P_f = \Pr(G(x) > 0) = \int_{G(x) > 0} I_f(x) f_G(x) dx = E[I_f(x)]
\]

where \( I_f(x) \) represents an indicator function as

\[
I_f(x) = \begin{cases} 
1, & \text{if } G(x) > 0 \\
0, & \text{otherwise} 
\end{cases}
\]

Then, we can use the expectation of the indicator function to represent the system reliability. However, when using MCS for reliability analysis, especially for high-reliability engineering system design, a large number of sample points needs to be generated which requires a large number of function evaluations and leads to a prohibitive computational cost. To alleviate this difficulty, the Kriging surrogate technique can be used as the performance function approximation.

### 2.3 Kriging-based Surrogate Model

The Kriging-based surrogate models have become one of the most popular methods to develop computationally efficient surrogate models in many engineering design applications, including simulation-based design optimization and uncertainty analysis. In this study, the Kriging method has been used to build surrogates to predict system performance at unobserved points. We developed our approach based on the ordinary Kriging model for surrogate model development. An ordinary Kriging model can be generally expressed as [9]

\[
G_k(x) = f(x) + S(x)
\]

where \( S(x) \) is a Gaussian stochastic process with zero mean and variance \( \sigma^2 \); the polynomial term \( f(x) \) can be replaced by a constant value, the mean response value \( \mu \), and \( G_k(x) \) represents the prediction result of performance function at point \( x \) using Kriging model.

\[
G_k(x) = \mu + S(x)
\]

In this Kriging model, the covariance function between arbitrary two input values \( x_i \) and \( x_j \) can be defined as

\[
\text{Cov}[S(x_i), S(x_j)] = \sigma^2 R(x_i, x_j)
\]

where \( R(x_i, x_j) \) denotes the correlation function matrix, and can be defined as

\[
R(x_i, x_j) = \text{Corr}(x_i, x_j) = \exp\left[-\sum_{\nu=1}^{n} a_{\nu} | x_i^\nu - x_j^\nu | ^{b_{\nu}} \right]
\]

where \( a_{\nu} \) and \( b_{\nu} \) are parameters of the Kriging model.

With \( n \) observations \( O = [X, G] \), where \( X \) is the input data set and \( G \) is the input performance, the log-likelihood function of the Kriging model can be expressed as

\[
\text{Likelihood} = -\frac{1}{2} \left[ n \ln(2\pi) + n \ln \sigma + \ln |R| + \frac{1}{2\sigma} (G - \mu) R^{-1} (G - \mu) \right]
\]

where \( A \) is an \( n \times 1 \) unit vector. Then \( \mu \) and \( \sigma^2 \) can be solved by maximizing the likelihood function, as

\[
\mu = [A^T R^{-1} A]^{-1} A^T R^{-1} G
\]

\[
\sigma^2 = (G - \mu)^T R^{-1} (G - \mu)
\]

With the Kriging model, the response for any given new point \( x' \) can be estimated as

\[
G_k(x') = \mu + r^T R^{-1} (G - \mu)
\]

where \( r \) is the correlation vector between \( x' \) and the sampled points \( X \). The mean square error \( e(x') \) can be computed by

\[
e(x') = \sigma^2 \left[ 1 - r^T R^{-1} r + \frac{(1 - A^T R^{-1} r)^2}{A^T R^{-1} A} \right]
\]

Thus, the prediction of response values at the new sample point \( x' \) using the Kriging model can be considered as a random variable that follows a normal distribution with mean \( G_k(x') \) and variance \( e(x') \).
3. Sensitivity Analysis Methods

Design sensitivity information is essential for RBDO in the iterative design processes, as it affects not only the accuracy of reliability analysis in each design iteration but also the convergence rate to the optimum design. This session introduces the developed adaptive sensitivity analysis technique, along with several existing sensitivity analysis methods employed in this study. These sensitivity analysis techniques will be compared with respect to the accuracy and efficiency of a numerical case study in the later section of the paper, in order to demonstrate the efficacy of the developed methodology.

3.1 The Finite Difference (FD) Method

The finite difference method has been widely used for design sensitivity analysis, with moving step size being set generally as a constant or a coefficient of design variables. In the finite difference method implementation, the approximation is inaccurate such that it sometimes leads to divergence, more iterations, and more computational cost in the design process. The main concept behind any finite difference scheme is related to the definition of the derivative of a smooth function \( u \) at a point \( x \in \mathbb{R} \). One-dimensional case can be expressed as

\[
u'(x) = \lim_{h \to 0} \frac{u(x + h) - u(x)}{h}
\]

where \( h \) should be sufficiently small to get a good approximation.

3.2 The Stochastic Sensitivity Analysis (SSA) Method

The stochastic sensitivity analysis method has been used to approximate smoothed sensitivity information of reliability with respect to the random variables based upon the approximation as

\[
R = P(G(x) > 0) = \int \cdots \int_{G(x) > 0} f_z(x) \, dx \approx \Phi \left( \frac{\mu(G(x))}{\sigma(G(x))} \right) \approx \Phi \left( \frac{\mu_{G(x)}}{\sigma_{G(x)}} \right)
\]

where \( \mu(G(x)) \) and \( \sigma(G(x)) \) are the mean and variance of performance function \( G(x) \) given certain distribution of the random input \( x \). The sensitivity of reliability with respect to the \( i \)th design variable \( d_i \) can then be approximated as

\[
\frac{\partial R}{\partial d_i} \approx \frac{\partial}{\partial d_i} \left[ \frac{\mu_{G(x)}}{\sigma_{G(x)}} \right] \times \text{pdf} \left[ \frac{\mu_{G(x)}}{\sigma_{G(x)}} \right]
\]

3.3 The Adaptive Finite Difference (AFD) Method

Since the new design point in the subsequent design iteration will be affected by the sensitivity estimation, inaccurate approximation might lead to divergence or more iterations. This paper, therefore, introduces an adaptive sensitivity analysis approach to accelerate the optimization process and reduce computational cost. From accuracy perspective, when the design points located close to each other meaning that the design points move with a small step size and probably approach to a local minimum, then a more accurate gradient information is needed. Then, the step size used in the finite difference method will be adjusted to obtain a better approximation.

To measure the distance between the current design point with the previous design points, we transform the design space into a standard normal space. Then the reliability index \( \beta \) becomes the distance corresponds to the target reliability called reliability index. The distance between the current design point and previous design points will be calculated and compared with the reliability index \( \beta \). When the distance between the current design point and the previous design points less than \( \beta \), more accurate gradient information is needed for the new moving direction. Also, with the increase of the iteration, design points moving closer to the optimal design, then a smaller step size need to be chosen. Our advanced technique will adjust this step size used in the finite difference calculation base on the trajectory of the design points, the reliability index, the moving distance and the number of iterations which can be defined as follows

\[
\text{step size} = \frac{\text{dist}_{\text{iteration}}}{K}
\]

where \( K \) is a constant number, here we choose \( K \) equals to 15. In this way, it enables us to determine when and where to reduce the moving step size to achieve the desired balance between the efficiency of the RBDO process and the accuracy of the reliability analysis during the intermediate design iterations.
4. Case Study and Results

In this section, one numerical example has been investigated to demonstrate the accuracy and efficiency as well as applicability of the proposed method. Through this numerical case study, different sensitivity analysis methods have been compared with respect to accuracy and efficiency. The case study considers the following mathematical design optimization problem with two random design variables $X_1$ and $X_2$. Both random variables are normally distributed as $X_1 \sim N(\mu_1, 0.3464^2)$ and $X_2 \sim N(\mu_2, 0.3464^2)$, where the design variable $d = [d_1, d_2]^T = [\mu(X_1), \mu(X_2)]^T$. The initial design point is set as $d_0 = [2, 1, 1, 59]$. The target reliability level is set to $R_t = 0.99$. The RBDO problem is formulated accordingly as shown in Eq. (18).

Minimize: \[ \text{Cost} = 10 - d_1 + d_2 \]
subject to: \[ P_i[G_i(X) < 0] \leq 1 - R_t, \ i = 1 \sim 3 \]
\[ 0 \leq d_1 \leq 10 \]
\[ G_i = 1 - \frac{X_1 X_2}{20} \]
\[ G_i = 1 - \frac{(X_1 + X_2 - 5)^2}{30} - \frac{(X_1 - X_2 - 12)^2}{120} \]
\[ G_i = 1 - \frac{80}{(X_1^2 + 8X_2 - 5)} \]

We use the distance between the final design and the optimal to measure the accuracy, and the number of iterations to measure the efficiency of the method. Four different approaches are compared: the finite difference method with 5% variation, FD with 10% variation, the smooth sensitivity analysis, and the adaptive finite difference approach. The results show in Table 1. The SSA and AFD methods have a relatively small mean and variance value in terms of both optimum error and number of iterations (shown as in Figure 1). The SSA and AFD have outperformed FD5 and FD10 in the presented case study.

![Figure 1. Accuracy and efficiency results of the case study: accuracy (left) and efficiency (right)](image)

<table>
<thead>
<tr>
<th>Optimizers</th>
<th>Optimum Error</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Var</td>
</tr>
<tr>
<td>------------</td>
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</tr>
<tr>
<td>FD5</td>
<td>1.7</td>
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</tr>
<tr>
<td>FD10</td>
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<tr>
<td>SSA</td>
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<td>0.009</td>
</tr>
<tr>
<td>AFD</td>
<td>1.63</td>
<td>0.31</td>
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</table>

5. Conclusions

This paper presented a new design sensitivity analysis technique for reliability-based design optimization employing surrogate models. The developed technique is based on the finite difference concept while efficiently and accurately combining the optimization process with the resolution of the sensitivity. The reliability index and the trajectory of the
optimization process have been used as quantitative metrics to adaptively adjust the step size for the finite difference method. The developed sensitivity analysis technique enables us to determine when and where to reduce the moving step size to achieve a desired balance between the efficiency of the RBDO process and the accuracy of the reliability analysis during the intermediate design iterations. The numerical case study of the reliability analysis and design demonstrated that the developed new design sensitivity analysis approach would improve both efficiency and accuracy of RBDO problems as compared to existing sensitivity analysis methods.

Acknowledgments

This research is partially supported by National Science Foundation (NSF) through Faculty Early Career Development (CAREER) award (CMMI-1351414) and the Award (CMMI-1538508), the NSF engineering research center of Power Optimization of Electro-Thermal Systems (POETS), and the University of Illinois at Urbana-Champaign.

References

RELIABILITY BASED ELASTO-PLASTIC TOPOLOGY OPTIMIZATION

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Abstract

This paper presents elasto-plastic topology optimization with a reliability constraint. It recalls fundamental concepts from first order reliability analysis and introduces an algorithm for topology optimization of elasto-plastic structures. The presented numerical example shows dependence of the volume fraction on the probability of failure.

Key words: Structural Topology Optimization, Elastoplastic Analysis, Reliability-Based Optimization

1. Introduction

An important aspect of any optimization process is robustness to the variability of structural parameters, either material or loading dependent. The more the structure becomes optimal, the lower the resistance to its parameter changes. One of the possible ways to tackle this issue is add to the optimization formulation an additional constraint for the probability of failure. The designer will then assure that his or her optimized structure does not go below the assumed safety level. Since the probability of failure of engineering structures must be small (about 0.001), it is possible to obtain a relatively fast estimation of reliability by using first or second order methods, i.e., First Order Reliability Method (FORM) or Second Order Reliability Method (SORM). Most often, several iterations (finite element solutions) are enough to obtain convergence. Recent advances in reliability based topology optimization have been presented in papers by Kharamanda et al. 2002 [1], Kim et al. 2009 [2], Kharamanda et al. 2014 [3], Kang and Liu 2018 [4], Chun et al. 2019 [5]. The authors of this study have also published several papers a this topic for more than a decade (Logo et al. 2011 [8], Błachowski et al. 2018 [9], Taugowski et al. 2019 [10]).

In the present paper iterative topology optimization algorithm together with elasto-plastic material formulation and reliability approach will be shortly described.

Effectiveness of the proposed methodology will be presented on a numerical example of elasto-plastic structure. All aspects of finite element computations, topology optimization as well as reliability analysis are performed using our own software implemented in MATLAB and C++ languages.

2. Theoretical Background on Reliability Analysis

Topological optimization allows to minimizing the mass of the structure while maintaining certain mechanical properties. By minimizing the mass, we also reduce the durability of the structure, in terms of random events. In order to maintain control over the reliability in the optimization process, it is necessary to assess also the safety level, together with mechanical properties. These properties include: loads, material or shape parameters. Vector of random parameters \( \mathbf{x} \) belongs to the probabilistic space \( \Omega \) (Figure 1). Some realizations of random variables \( \mathbf{x} = \{x_1, \ldots, x_N\} \) may lead to a topology that does not meet the constraints. Such structures are in a state of failure. Constraints therefore divide the probabilistic space into two domains: the safe domain \( \Omega_s \) and the failure domain \( \Omega_f \) (Figure 2). To check in which state given random realization is, limit state function \( g(\mathbf{x}) \) is introduced. Similarly to the constraints in deterministic optimization there is a convention assuming that a negative value of function \( g(\mathbf{x}) \) means that \( \mathbf{x} \) is in failure domain while a positive value means safe domain of \( \mathbf{x} \).

\[
g(\mathbf{x} \in \Omega_s) > 0 \text{ safe domain}
\]
\[
g(\mathbf{x} \in \Omega_f) < 0 \text{ failure domain}
\]
Equation $g(x) = 0$ determines limit state surface which separates safe and failure domains (Figure 1). The probability of failure can be computed as the integration of joint probability distribution over failure domain. It is expressed by the following formula:

$$P_f [x \in \Omega_f] = P[g(x) < 0] = \int_{g(x) < 0} f_x(x) \, dx$$  \hspace{1cm} (1)

Direct computation of the probability of failure using above formula is very difficult or even impossible due to an implicit form of a function $g(x)$, which in the case of topology optimization, is based on finite element computations. There are several methods to determine the probability of failure $P_f$. Most versatile one: Monte Carlo (MC) method, allows to assess $P_f$ for any function continuous or discontinuous. Most serious drawback of MC method is its numerical complexity. Therefore, combining this method with very complex topological optimization is not an option. Another method: Importance Sampling is less complex, but still it takes an unacceptably long time to determine the probability of failure. Most promising method seems to be First Order Reliability Method (FORM).

The foundations for modern methods of reliability analysis were developed by Hasofer and Lind [6]. In their work the concept of the so-called design point or most probable point was introduced.

Design point is the realization of random variables with the most probable failure scenario. In other words, it is a point on the limit surface in which the joint probability density function reaches its maximum value (Figure 2). Therefore, the neighborhood of this point has the greatest contribution in determining the integral (1). Hence, in the case of a small probability of failure, the approximation of the limit surface by the plane will result an acceptable estimation of the probability of failure. In topological optimization, as in the case of engineering structures, the probability of failure should be very small, usually about 0.001. What makes the FORM method a good choice for coupling with topological optimization.

Rackwitz and Fiesler[7] were the Authors of the first iterative gradient based algorithm for searching the design point. At the beginning, to facilitate computations, probabilistic design space is transformed to dimensionless standard normal space $x \rightarrow u$. The design point is the closest point of limit state surface to the center of the coordinate system of standard normal space. So its determination is in fact an optimization task formulated as follows:

Find $\min \| u' \| = u \cdot u^T$

constrains $g(u) = 0$  \hspace{1cm} (2)

The iterative Rackwitz–Fiesler formula has the following form:

$$u^{(\omega+1)} = \frac{1}{\| \nabla g(u^{(\omega)}) \|} \left( \nabla g(u^{(\omega)})^T u^{(\omega)} - g(u^{(\omega)}) \right) \nabla g(u^{(\omega)})$$  \hspace{1cm} (4)

Having a design point $u^{\ast}$ it is easy to use a linear estimation of the probability of failure. We assume that the approximate limit surface is a hyperplane and is tangent to the limit surface at the design point (Figure 2). Thus, the hyperplane equation has the form:

$$l(u) = -au + \beta$$  \hspace{1cm} (5)
where $a$ is a unit vector with the direction opposite to the gradient $\nabla g(u)$ at the design point $u^*$

$$a = \left. \frac{\nabla g(u)}{\| \nabla g(u) \|} \right|_{u = u^*}. \tag{6}$$

and $\beta$ is an Hasofer-Lind’s reliability index. The linear estimation of the probability of failure has the form:

$$P_{\text{FORM}} = \Phi(-\beta) \tag{7}$$

$$\beta_{\text{FORM}} = \text{sign} (g(0)) \| u^* \|. \tag{8}$$

3. Topology Optimization Approach

In the case of topology optimization vector of random variables $x$ can be composed of loads or material constants. Shape of the structure is a result of topology optimization therefore random nature of shape parameter is not taken into consideration. Now we present complete topology optimization with reliability constrains in the form of algorithm and flowchart.

<table>
<thead>
<tr>
<th>Algorithm 2, Topology optimization of elasto-plastic structures with reliability constrains.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1. Initialize design variables to a vector of ones $\rho^{(0)} = {1,1,\ldots,1}$ and erased element list to an empty list $L = {}$. Initialize random variables vector $x = {x_1, x_2, \ldots, x_N}$. Until the load capacity or displacements condition is exceeded repeat Steps 2 to 7.</td>
</tr>
</tbody>
</table>

Step 2. At every $k$-th iteration solve nonlinear equilibrium equations for the elasto-plastic problem

$$K(\rho^{(k)})u(\rho^{(k)}) = f = 0,$$

Step 3. Determine the stress intensity vector calculated as the average of equivalent von Mises stresses evaluated at each Gauss point, then normalize the obtained values dividing them by the yield limit

$$\sigma = \frac{1}{N_x} \sum_{x=1}^{N_x} \sigma_x, \quad i = 1,2,\ldots,N.$$

Step 4. Apply a design filter to avoid the checkerboard phenomenon

$$\{\sigma\}_{\text{filter}} = \rho^{(k)} \sum_{j=1}^{N} H_i \rho_j^{(k)} \sigma_j,$$

where $H_i$ is the convolution operator $H_i = r_{\text{min}} - \| x_i - x_j \|$ and $j$ belongs to the set of elements for which the distance from the center of the $i$-th element is smaller than the filter radius $r_{\text{min}}$.

Step 5. Perform FORM to assess probability of failure:

$$P_{\text{FORM}} = \Phi(-\beta)$$

If structure is not reliable ($P_{\text{FORM}} > P_{\text{limit}}$) then stop.

Step 6. Select $n$ finite elements with the smallest stress intensities $\sigma_i < \sigma_{\text{min}} + \sigma_i$ (usually $\sigma_i = 0.005$) and add the list of the newly selected elements $l$ to the list of previously erased elements, $L^{(k)} = \{L^{(k-1)} \cup l\}$.

Step 7. Using the current list of erased elements $L$ update corresponding design variables applying the following iterative formula:

$$\rho^{(k)} = \max_{l \in L} \rho_{\text{max}} \left[ \{\sigma\}_{\text{filter}} \right]^{(k-1)}.$$
4. Numerical Example

Reliability assessment in topological optimization will be illustrated on a simple benchmark presented in Figure 3. The example deals with a cantilever with one force applied at its free end. Regular rectangular mesh composed of Lagrange four-node finite elements were used in the example. Mesh dimension is 40 x 20 elements.

Aluminum was chosen as the material with the following parameters: Young’s modulus $E = 210 \text{GPa}$, Poisson’s ratio $\nu = 0.33$, yield stress $\sigma_y = 460 \text{MPa}$, thickness $h = 0.22 \text{units}$.

Figure 3. Design domain for cantilever under investigation

Flowchart of the proposed method for topology optimization under reliability constraint.
Figure 4. Optimal topology with reliability constraints $P_r = 0.005$.

Probabilistic nature is represented by three random variables Young’s modulus, yield stress and Poisson’s ratio. Distribution parameters of random variables displayed in Table 1.

<table>
<thead>
<tr>
<th>variable</th>
<th>mean value</th>
<th>standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
<td>210 GPa</td>
<td>10%</td>
</tr>
<tr>
<td>yield stress</td>
<td>460 MPa</td>
<td>10%</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.33</td>
<td>10%</td>
</tr>
</tbody>
</table>

Limit state function reflects displacement condition. Unsafe state means the displacement in point $c$ (see Figure 3) $u_c$ exceeds permissible value 0.1 m ($l/200$), so the limit state function is specified by the formula:

$$g(u) = 0.1 - u$$

(9)

In Figure 5, dependence of the volume fraction on the probability of failure has been presented. An acceleration in the increase in the probability of failure was observed for $V_w \approx 42\%$. At this point, yield surface was reached in fixed vertices what increases the deformation process and value of displacement.

Figure 5. Probability of failure as a function of volume fraction.

5. Conclusions

In the present paper, topology optimization with reliability constraint was formulated. The final topology of a cantilever with reliability constraint was shown in Figure 4. Additionally, in Figure 5, one can observe that above certain level of volume fraction $V_w \approx 42\%$, the probability of failure rapidly grows. This phenomenon is caused by the start of the plasticization process.
Acknowledgments

The present study was supported by the National Research, Development and Innovation Office (Grant K 119440) and by the joint grant of the Hungarian and the Polish Academy of Sciences.

References


STRUCTURAL OPTIMIZATION CONSIDERING DYNAMIC RELIABILITY CONSTRAINTS VIA PROBABILITY DENSITY EVOLUTION METHOD

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Abstract

In this paper, an approach for dynamic reliability-based design optimization (DRBDO) problems is presented. In this approach, the probability density evolution method (PDEM) is employed as a highly efficient reliability analysis method to handle reliability constraints of dynamical systems. In order to enhance the efficiency of gradients estimation, the change of probability measure, a strategy based on the Radon-Nikodym derivative is introduced. By synthesizing the change of probability measure and the PDEM, the gradients of failure probability constraints can be evaluated without any extra performance function evaluation. Then, to solve DRBDO problems, this reliability and gradient evaluation framework is embedded into the sequential approximate programming method where the gradient information is used to generate the explicit approximation of failure probability constraints. Finally, a numerical example is presented to demonstrate the application and advantage of the proposed method. The results of the numerical example indicate that the proposed method is effective, especially for the problems in which the design variables are distribution parameters of the probability density functions that characterize the random variables.

Key words: Reliability-Based Design Optimization, Dynamic Reliability, Probability Density Evolution Method, Change of Probability Measure

1. Introduction

Structural optimization is considered as an effective approach to the tradeoff between structural performance and cost. However, uncertainties in structural material properties, geometry dimensions and loading conditions always lead to significant fluctuations in structural performance [1]. Therefore, reliability constraints should be imposed in the optimization process to ensure the safety of structures.

During the past three decades, many methods have been developed to solve reliability-based design optimization (RBDO) problems [2-3]. Nevertheless, these methods mainly aim at problems with respect to reliability constraints of static systems. For civil engineering structures, dynamic excitations, e.g., earthquakes, winds and sea waves, must be taken into consideration. Consequently, it is necessary to perform dynamic reliability-based design optimization (DRBDO). This will of course be much more involved due to the conceptual and numerical complexity in dynamic reliability evaluations of structures. In this framework, some methods have been suggested and applied to a number of complex problems involving stochastic excitation and deterministic design variables [4-5].

On the other hand, gradient-based optimization methods are widely used to solve structural optimization problems due to their fast convergence. In this class of methods, gradients of objective and constraint functions with respect to design variables have to be evaluated frequently. It is noteworthy, however, that the estimation of gradients via finite difference will become quite inefficient when reliability constraints, especially dynamic reliability constraints, are involved in the problem [6].

In recent years, the probability density evolution method (PDEM), where a completely state-decoupled generalized density evolution equation (GDEE) is proposed, has been shown to be a very efficient reliability evaluation method [7]. At the same time, the nature of PDEM in the sense that evaluations of structural performance and GDEE are totally decoupled provides an efficient and accurate approach for the estimation of gradients. In the present work,
PDEM is embedded into the optimization process to solve DRBDO problems.

Moreover, a strategy based on the Radon-Nikodym derivative is introduced as a powerful and efficient tool to estimate the gradients of failure probabilities [8]. Finally, a numerical example is presented to demonstrate the application and advantage of the proposed method. The results of numerical example indicate that the proposed method is efficient and effective, especially for the problems in which the design variables are distribution parameters such as mean values of random variables.

2. Formulation of Dynamic Reliability-based Design Optimization

There are different formulations available of RBDO and DRBDO problems [9-10]. For example, minimization of construction costs under reliability constraints or failure probability constraints is one of the most common formulations used in the area of structural optimization. This type of formulation is adopted in the present paper. Thereby, the optimization model of DRBDO problems is defined as

\[
\begin{align*}
\min & \quad E[f(x, \Theta)] \\
\text{s.t.} & \quad F_i(x, \Theta) \leq F^i_j; i = 1, 2, \ldots, m \\
& \quad x_i \leq x_i \leq x^i_i; i = 1, 2, \ldots, n
\end{align*}
\]

where \( x = (x_1, x_2, \ldots, x_n)^T \) denotes the \( n \)-dimensional vector of design variables; \( \Theta = (\Theta_1, \Theta_2, \ldots, \Theta_k)^T \) denotes \( k \)-dimensional vector of random variables; \( f(x, \Theta) \) is the objective or cost function; \( F_i(x, \Theta) \) is the failure probability of \( j \)-th failure mode at design point \( x \); \( F^i_j \) is the prescribed target failure probability of \( j \)-th failure mode; \( x \) and \( x^i \) are the prescribed lower and upper bounds of design variables \( x \), respectively.

In this work, a class of DRBDO problems are considered where design variables are defined as the mean values of random variables. The random vector \( \Theta \) is assumed to be \( \Theta = x + \varepsilon \), where \( \varepsilon \) is a vector of zero-mean random variables. Noting that design variables \( x \) will be changed within each optimization iteration while \( \varepsilon \) is left invariant, the DRBDO problem is reformulated as

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad F_i(x) \leq F^i_j; i = 1, 2, \ldots, m \\
& \quad x_i \leq x_i \leq x^i_i; i = 1, 2, \ldots, n
\end{align*}
\]

where the cost function has been assumed to be linear. The research in this paper is devoted to solve DRBDO problems formulated as in Eq. (2).

3. Reliability Assessment and Gradient Evaluation

3.1 Fundamentals of Probability Density Evolution Method (PDEM)

In this paper, PDEM is employed to assess the dynamic reliability of structures. Thus, a brief introduction of the PDEM [7] is firstly outlined in this section.

Generally, consider a dynamical system with random parameters

\[
\dot{Z}(t) = G(Z(t), \Theta, t); Z(0) = z_0
\]

where \( Z = (Z_1, Z_2, \ldots, Z_N)^T \) is the \( N \)-dimensional state vector; \( z_0 \) denotes the \( N \)-dimensional initial vector; \( G = (G_1, G_2, \ldots, G_N)^T \) is a vector of \( N \) functions; \( \Theta = (\Theta_1, \Theta_2, \ldots, \Theta_k)^T \) is the \( k \)-dimensional random vector with the joint PDF \( p_\Theta(\Theta) \). For a well-posed dynamical system, Eq. (3) is uniquely solvable, and the solution relies on the initial condition, random vector and time. Therefore, the solution can be assumed to take the form

\[
Z(t) = H(z_0, \Theta, t) \quad \text{or} \quad Z_i(t) = H_i(z_0, \Theta, t); i = 1, 2, \ldots, N
\]

Due to the fact that all source of randomness is characterized by the random vector \( \Theta \), the augmented system \((Z(t), \Theta)\) is a probability preserved system. Based on the principle of preservation of probability, the GDEE is derived as follows

\[
\frac{\partial p_{\Theta}(z, \Theta, t)}{\partial t} + \sum_{i=1}^{N} \frac{\partial p_{\Theta}(z, \Theta, t)}{\partial z_i} \frac{\partial H_i(z_0, \Theta, t)}{\partial z_i} = 0
\]

where \( p_{\Theta}(z, \Theta, t) \) is the joint PDF of \( Z(t) \) and \( \Theta \). The initial condition for Eq. (5) is
\[ p_{z_0} (z, \theta, t) = \delta(z - z_0) p_{\theta} (\theta) \]  

If only one physical quantity is considered, the GDEE is reduced to

\[ \frac{\partial p_{z_0} (z, \theta, t)}{\partial t} + \dot{Z}(\theta, t) \frac{\partial p_{z_0} (z, \theta, t)}{\partial z} = 0 \]  

Noting that it is hard to get \( \dot{Z}(\theta, t) \) in closed form, a numerical method to solve Eq. (5) or Eq. (7) was proposed in [1]. In general, four steps are involved in solving the GDEE:

i. Discretize the probability space \( \Omega_p \) by an optimal point selection strategy [11]. The selected point set is denoted by \( M = \{ \theta_q, P_q \} | q = 1, 2, \ldots, n_m \} \), where \( \theta_q \) is the \( q \)-th point of a total of \( n_m \) representative points with respect to the PDF \( p_{\theta} (\theta) \); \( P_q \) is the assigned probability of \( \theta_q \).

ii. For each representative point \( \theta_q \), carry out the deterministic analysis to acquire the velocity response \( \dot{Z}(\theta_q, t) \) of the system.

iii. For each representative point \( \theta_q \), introduce \( \dot{Z}(\theta_q, t) \), into Eq. (7) and solve the equation by the finite difference method with total variation diminishing (TVD) scheme.

iv. Get the PDF results by numerical integration with respect to representative points:

\[ p_{z_0}(z, \theta, t) = \int_{\Omega_p} p_{z_0}(z, \theta, t) \, d\theta = \sum_{q=1}^{n_m} p_{z_0}(z, \theta_q, t) \]  

By adopting the method of absorbing boundary conditions [12] or the idea of equivalent extreme-value event [13] into the framework of PDEM, the reliability and failure probability can be assessed.

### 3.2 Change of Probability Measure

To evaluate the gradient of failure probability, the strategy of change of probability measure [8] is considered in this work. This strategy will remarkably decrease the time for evaluating the gradients when combining with PDEM. The strategy of change of probability measure is built on the rigorous basis of Radon-Nikodym theorem. When introduced into PDEM, the change of probability measure is nothing but recalculation of assigned probabilities with respect to the selected point set.

In order to evaluate the gradient by the change of probability measure and the PDEM, four steps are needed:

i. A full PDEM is performed at the current design point \( x^{(i)} \) to acquire the failure probability \( F_i(x^{(i)}) \).

ii. Select a proper step length \( \Delta x \), for finite difference, and a renewed joint distribution \( p_{\theta} (\theta) \) with respect to the mean-value vector \( x + \Delta x, \epsilon \) is derived, where \( \epsilon \) is the \( i \)-th standard basis. Then, the assigned probabilities of the representative point set in Step i, i.e. \( M = \{ \theta_q, P_q \} | q = 1, 2, \ldots, n_m \} \), is updated according to the renewed PDF \( p_{\theta} (\theta) \). The updated representative point set is denoted by \( \tilde{M} = \{ \tilde{\theta}_q, \tilde{P}_q \} | q = 1, 2, \ldots, n_m \} \).

iii. For each representative point \( \tilde{\theta}_q \), solve the GDEE again with the velocity response \( \dot{Z}(\tilde{\theta}_q, t) \) collected in Step i and the updated assigned probability \( \tilde{P}_q \). Then, the failure probability \( F_i(x, x + \Delta x, \epsilon) \) is obtained.

iv. The gradient of failure probability with respect to design variable \( x \) at \( x^{(i)} \) can be approximated by

\[ \frac{\partial F_i}{\partial x_i} \bigg|_{x=x^{(i)}} = \frac{F_i(x + \Delta x, \epsilon) - F_i(x^{(i)})}{\Delta x} \]

Eq. (9) estimates the gradient by the forward scheme. In fact, the backward and central scheme can also be employed to evaluate gradient of failure probability in the proposed method.

It is noteworthy that by means of the PDEM and change of probability measure, no extra deterministic response analysis is required in gradients approximation. Thereby, the proposed method is especially applicable for DRBDO problems in which deterministic response analyses are expensive.

The proposed gradient estimation method can be combined with any gradient-based optimization algorithm. In this paper, the sequential approximate programming algorithm proposed in [14] is employed. This algorithm is based on convex linearization [15], diagonal quadratic approximations [16], and conservative convex and separable approximations [17]. The optimal point of each sub-problem is feasible if the initial point is located in the feasible design space.
4. Numerical Example

In this section, the dynamic reliability-based stiffness optimization of a 2-story frame under earthquake excitation is considered to show the efficiency, validity and effectiveness of the proposed method.

\[
\dot{\gamma}(t) = \ddot{u}(t) - \dot{u}(t) \left[ H(\tau(t) - \tau_p) \frac{\tau(t) - \tau_p}{\tau_p - \tau_p} H(\tau - \tau(t)) + H(\tau(t) - \tau_p) \right]
\]

where \( \tau(t) = u(t) - \gamma(t) + \gamma(t) \); \( u(t) \) is the inter-story drift history; \( \gamma(t) \) and \( \gamma(t) \) denote the plastic elongations of the device. The parameters are set equal to \( \tau_p = 0.006 \text{m} \), \( \tau_p = 0.0042 \text{m} \) and \( k_p = 6,0 \times 10^4 \text{N/m} \) in this case. A typical restoring force curve of the devices is shown in Figure 2, indicating a strong nonlinearity in the structural response.

![Typical restoring force curve of the nonlinear devices](image)

Considering the uncertainties in construction and material, the lateral stiffnesses are assumed to be independent normal random variables with coefficient of variation of 5%. The mean values are chosen as design variables. The objective is to minimize the total stiffness while the failure probability constraint is defined in terms of the maximum absolute inter-story drift. Therefore, the reliability constraint is defined as

\[
\text{Pr}(\max_{j=1}^{\text{max}} \left[ u_j (x, y, t) \right] > d^b) \leq P^b
\]

where \( T \) is the duration of the excitation; \( u_j \) is the \( j \)-th inter-story drift; \( d^b \) is the threshold for the maximum absolute inter-story drift and is set equal to 0.03m; \( P^b \) is the threshold for failure probability. Considering the lower and upper bound constraints of the design variables, the DRBDO problem is written as
\[
\begin{align*}
\min \quad & f(x_1, x_2) = x_1 + x_2 \\
\text{s. t.} \quad & \Pr \left\{ \max_{i=1,2} \max_{0 < \xi_i \leq 1} \left[ u(i, x_1, x_2, t) \right] \right\} \geq d^b \leq P^b \\
50 \times 10^4 & \leq x_1, x_2 \leq 110 \times 10^4
\end{align*}
\] (14)

Figure 3. Iteration history of design variables and objective values

The initial feasible design point is set equal to \((110, 110) \times 10^4 \text{ N/m}\). 200 representative points are selected to perform the PDEM and the change of probability measure at the current design point during the optimization iterations. The proposed method is employed to solve the DRBDO problem with different levels of \(P^b\), e.g., 0.05, 0.01 and 0.001, respectively. The iteration history of the design variables and objective values is shown in Figure 3. It is evident that the optimization process converges to the optimal design point within a few iterations. In view of the fact that only one full PDEM is performed at each iteration, no more than 2,000 deterministic response analyses are carried out during the optimization process for each case. In order to check the correctness of the results, the optimal design points with different \(P^b\) are plotted in Figure 4, together with the contours of failure probability. It is noteworthy that there exist remarkable differences between the optimal design obtained by the deterministic optimization problem and the DRBDO problems. This demonstrates that the explicit consideration of uncertainty in structural design optimization problems is of great significance.

Figure 4. Optimal design points in feasible region

5. Conclusion

A method based on the probability density evolution method for a dynamic reliability-based design optimization problem has been proposed. The change of probability measure is adopted to estimate the gradients of failure probability without any extra deterministic response analysis. Based on a 2-story frame structure with nonlinear devices under earthquake excitation, a dynamic reliability-based design optimization problem is solved as a numerical example. The validity and efficiency of the proposed method for DRBDO problems, where design variables are regarded as mean values of random variables, are demonstrated by the results of the example problem.
Acknowledgments

Financial support from the National Natural Science Foundation of China (NSFC Grant Nos. 51725804, 51538010, and 11761131014) and the NSFC-Guangdong Province Joint Project (Grant No. U1711284) is gratefully appreciated.

References


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UNDERSTANDING ROBUST DESIGN SPACES USING
USING SELF ORGANIZING MAPS

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Abstract

Metamodels are widely used to alleviate the computational expense involved in solving optimization formulations with expensive computer models. Metamodels are built based on the responses evaluated at Design of Experiment (DoE) points. Hence, the DoE and the number of points in the DoE play a vital role in the quality of the metamodel. Understanding function non-linearity is essential to decide the number of points required in the DoE which in turn governs the order of the metamodel that can be built. Self organizing maps (SOM) are predominantly used as visualization tools and for clustering. Here we propose using it for understanding correlation and nonlinearity in the design space, with small samples. The advantage of SOM is that the relationship between the response and any design variable is reduced to a two-dimensional plot allowing for easy visualization. This is extremely useful in high dimensional problem where nested axis plots and tile plots are not adequate. Further we propose to use SOM-based design space exploration to identify zones of interest for robust design. Robust design refers to small response variation regardless of the intensity of variation in design variables, which often translates to flat or small gradient regions in the response function. Based on limited input samples, we use SOM to explore design space and understand the flat regions.

Key words: Robust Design, Design Space Exploration, Self Organizing Maps

1. Introduction

Design Space Exploration (DSE) is a process in which design alternatives are explored for a variety of purposes such as finding optimal designs, identifying regions of interest, sampling for DoE, dimension reduction etc. DSE is widely used in engineering design tasks to understand complex design spaces or to decompose the design space. DSE might refer to: performing sensitivity studies to understand factor effects or response function non-linearity, creating nested plots to visualise response function variation in design space, conducting factor analysis to estimate factor effects or building surrogates to identify regions of interest in design space. The bottom line is to understand the non-linearity of response function in design space and capture it as accurately as possible. However, all these methods suffer from curse of dimensionality and the number of samples required increases exponentially with the dimensions. Hence, it is desirable to develop approaches that can allow designers to visualise function non-linearity in any dimension. Self Organising Maps (SOM) [1] is an unsupervised neural network, which is generally used for data mapping and currently being successfully used in data mining and visualization. SOM helps in representing multidimensional data onto the 2-dimensional grid, SOM can map non-linear relationship within multi-dimensional data into the simple geometric relationship among their nodes in the 2-dimensional grid. [2] used SOM along with fuzzy clustering to reduce design space into relatively smaller promising regions and performed optimization in the zones of interest. They analyze the relation between design variables and objective function using SOM built on initial samples followed by resampling in reduced design space. [3] used rough set theory and SOM to analyze optima in the design space. [4] explored adaptive sampling methods using SOM to identify set of input variables which would give desired responses. [5] used SOM to visualize pure data of robust design optimization to visualize spread of the data and analyze tradeoffs between multiple objectives.

Designers usually seek robust designs than optimal designs in the presence of uncertainties. Robust designs guarantee performance in spite of uncertainties. That is, the variability of output response is between some allowable threshold
in spite of variabilities in the input space. Robust region is usually characterized by flat regions in design space. Hence, it is desirable to capture flat regions in design space because they are likely to correspond to robust designs. It is to be noted that there may be multiple flat regions in design space. Our interest lies in understanding the function non-linearity in design space with small samples and further identify the region of interest for Robust Design Optimization (RDO) using SOM. The rest of the paper is organised as follows; section 2 discusses the concept and algorithm for SOM, robust design concepts are discussed in section 3 along with discussion on implementation of SOM for identifying regions corresponding to robust design followed by conclusions in section 4.

2. Self Organizing Maps

SOM is an unsupervised learning algorithm that provides visualization of high dimensional data by reducing it to the two-dimensional map. SOM is easy to understand. It can be used for clustering, prediction, data representation. The main advantage of SOM is that it preserves the topology of high dimensional data. SOM is similar to nonlinear Principal Component Analysis (PCA) but with the advantage of visualization. SOM differs from ANN on that fact that it does not have error-based learning. SOM works on competitive learning methodology. Once input data is presented to SOM, only one node which is closest to the input point will win. For every input data point, the nodes will compete with each other and the SOM grid gets trained. During the training, the values for the input variables are gradually adjusted to preserve neighborhood relationships (thereby capturing the topology) that exist within the input space. SOM has nodes at output layer arranged in rectangular or hexagonal lattice. SOM output has multiple components such as U-matrix, component planes, hit rate. 

For the distances between samples in design space. Component plane plots allow to understand the variation in each input variable and the output response. The algorithm for generating SOM is discussed next.

One starts with a collection of combination of input variables and the respective responses that form the dataset. Each row in a dataset corresponds to a design point and the respective response. The idea of SOM is to use a two dimensional grid of nodes to map each row of the dataset based on a distance criteria. SOM is a two dimensional array of nodes (neurons) where each node is characterised by a vector of weights. Weights here refers to the coordinate of the output node. The number of weights each output node will carry corresponds to the size of the dataset. The next step is, for each row of the dataset, find the node that is closest to it by computing the Euclidean distance between the dataset and the output nodes as presented in Eq. 1.

\[ D = \sqrt{(x_i - w_i)^2} \]  

where \( D \) is the euclidean distance, \( x_i \) is a row of the dataset, \( w_i \) are weight vector of the output node. The output node that has the least \( D \) is considered the Best Matching Unit (BMU) for the dataset row under consideration.

\[ v_i = [x_i, y], i \in [1, 2, 3, \ldots, d] \]
\[ w_i = [m_{i1}, m_{i2}, m_{i3}, \ldots, m_{id+1}] \]

where \( d \) is number of input variables in the problem and \( y \) is the output variable (which can also be multiple) and \( m_i \) are the weights, \( m_{i+1} \) is weight corresponding to the output variable. The radius of the neighborhood from BMU is to be found at every iteration and all the nodes in that radius will have their weights updated using the updation rule according to the equations 2, 3, 4.

\[ w_i(t+1) = w_i(t) + \varphi(t)L(t)(v_i(t) - w_i(t)) \]
\[ L(t) = \frac{L_o}{\lambda^t} \]
\[ \varphi(t) = e^{-\frac{t}{R_t}} \]

\( \varphi(t) \) is to ensure the weight change is more when the distance of updating node is less from BMU. The radius decays with iterations as in Eq. 5

\[ R(t) = R_o(t) \times \lambda^t \]

where \( R_o \) is maximum distance in Lattice and \( \lambda \) is the ratio of the total number of iterations to the \( R_o \). This process is repeated for all input vectors. SOM is demonstrated on two examples in the following. The focus here is to use SOM output to understand the non-linearity of the response function in the design space. Among other things, this information is useful in deciding the number of points required in a DoE which in turn affects the metamodel that can
be built, and its accuracy. We use the SOM toolbox for the implementation.

2.1 Understanding the Non-linearity of the Response Function in Design Space

Consider the function in Eq. 6. Here, we pretend that we have no information about the function. A DoE with 20 LHS points and respective response values are considered. SOM is trained over the dataset.

\[ z = x^2 + y^2, \quad x \in [-5, 5] \]  

400 output nodes are considered. Figure 1 (a) shows the trained SOM over the original surface given in Eq. 6. Figure 1 (b) shows the U-matrix and component plane plots. It can be observed from z component plane plot that the biquadratic nature of the function is captured. It can be seen that as \( x \) increases from top to bottom in the component plane \( z \) first decreases and then increases, which is similar to the original function. From the \( y \) component plane, it can be observed that as \( y \) is increasing from left to right, the \( z \) first decreases and then increases. In addition, the \( z \) component plane shows increasing trend as we go in radially outward direction. This exercise demonstrates that SOM can be used to understand the non-linearity of the response function in design space with respect to the design variables.

Figure 1. (a) SOM net over surface (b) U-matrix and component plane plots

2.2 Modified SOM Algorithm

Another instance of 20 LHS samples is considered. Figure 2 shows the trained SOM, It can be observed that the net is self intersecting. This is referred to as aberration [6] and leads to errors in predictions or understanding the function. With less than 2 dimension one can plot the SOM net to check for self intersection. For higher dimensions the aberration is evident from the component plane. That is, the component plane of the input dimensions is not ordered if there is aberration. If there is uniform increase or decrease in all input variable component plane in any one particular direction we can say that net is not self intersecting. In the event of self intersection, there should be abrupt change in the color of component plane it can be concluded that SOM net is self intersecting. For example, in Figure 2b, x component plane has abrupt change in trend. x should increase from left to right but it is first increasing then decreasing. Hence, one can suspect self intersection of net. Smoother maps tend to convey more information from the same data. Since, SOM is usually used to extract information on correlation or factor analysis, self intersection of the net was immaterial though it affects the topology preservation quality of self organizing map. However, in this work we have a different objective on understanding the relationship between the input variables and response, flatness etc and we seek to avoid self intersection of SOMs. We use an algorithm proposed in [7], where the BMU is found with help of input variables only. Response vector or output is used for updating only. This ensures that there will no intersection and folding in map. The results from the modified algorithm are discussed below:

\[ z = -20 \exp\left(-0.2 \sqrt{\left(\frac{1}{5} \sum_{i=1}^{5} x_i^2\right)} - \exp\left(\frac{1}{5} \sum_{i=1}^{5} \cos(2\pi x_i) + 20 + \exp(1)\right)\right), \quad x_i \in [-1, 1] \]  

(7)

Ackley function with 5 input variables and one output as shown in Eq. 7 is considered to demonstrate the modified
SOM algorithm. 20 LHS data point is considered and SOM net is trained as per the modified algorithm. Figure 3 shows the U-matrix and component plane after training. It can be seen from the $z$ component plane that there is valley around the centre of the $z$ component plane and the value increase as we travel radially outwards from the valley. We can get the region of interest by values of weights associated with corresponding areas of interest. Examining the component planes, one can conclude how the response function varies with respect to the input variables providing a sense of sensitivity information.

![Figure 2](image.png)

Figure 2. (a) Folded SOM net over surface (b) U-matrix and component planes

The point to be noted in the modified algorithm is that for a fixed DoE, the trend of component plane plots is preserved. This is not guaranteed in the conventional algorithm.

Figure 4 shows the U-matrix and component plane obtained by the conventional SOM algorithm. It can be clearly seen that there is no uniform trend in the component plane hinting the self intersection and folding in the trained SOM net.

3. Robust Design

Uncertainties are inevitable in design and it causes poor performance of product. These uncertainties can be caused by variation in material properties, load conditions etc. This uncertainty then gets propagated to output. Traditional design approach takes factor of safety into account and seeks a conservative design. To handle these uncertainties in better way one can use probabilistic techniques. Robust Design Optimization (RDO) is one of the ways to deal with this without removing source of uncertainties. As shown in Figure 5, design optimization will choose A as solution but even slight variation in the input $x$ causes a large variation in output. Whereas RDO will choose B as output where if there is variation in input $x$ will cause lesser variation in output as compared to the A. The characteristic of the region corresponding to robust design is flat or near flat with less gradient. Hence, in RDO, it is advantageous to know flat regions apriori so that more samples can be used in the flat region thereby increasing the chances of identifying a robust optima. We propose using modified SOM to visualize the design space and identify flat regions.

As discussed above, SOM is obtained based on the DoE points. Once the weights are obtained, K Means algorithm is used for clustering. Number of clusters is decided based on Davies-Boulding index. For each cluster the standard deviation of the weights of the response is computed. When the standard deviation is less, it is likely to point to flat regions or near flat regions leading to robust regions in design space.

Results are validated by comparing trends in standard deviations of responses calculated using original function on large number of samples sampled in each of the clusters. Standard deviation from original function evaluations is denoted by $\sigma$, and $\sigma_k$ denote standard deviations from the response coordinate of weight vectors belong to respective clusters.
Figure 3. U-matrix and component plane plots using modified SOM algorithm

Figure 4. U-matrix and component plane using conventional algorithm
Consider the Aspenberg function [7] in Figure 6 and given by Eq. 8.

\[ y = 4x^2 + e^{-30x^2} + e^{-10(x-0.25)^2} \]  

Figure 6 (a) shows there exist deterministic optima in region where \( x \in [-0.4,0] \) and robust region lies in region \( x \in [0,0.25,0.5] \). SOM net is obtained based on 15 LHS samples.

**Table 1. Summary of Cluster Statistics**

<table>
<thead>
<tr>
<th>Cluster</th>
<th>( \sigma_i )</th>
<th>( \sigma_j )</th>
<th>( x_{\text{min}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cluster 1</td>
<td>0.1684</td>
<td>0.1064</td>
<td>[0.0596, 0.6358]</td>
</tr>
<tr>
<td>Cluster 2</td>
<td>0.4424</td>
<td>0.5797</td>
<td>[0.6767, 1.0039]</td>
</tr>
<tr>
<td>Cluster 3</td>
<td>0.4008</td>
<td>0.3763</td>
<td>[-0.6732, -0.1005]</td>
</tr>
<tr>
<td>Cluster 4</td>
<td>0.4607</td>
<td>0.6022</td>
<td>[-1.0004, 0.7141]</td>
</tr>
</tbody>
</table>

Proposed approach gives 4 clusters as output. Standard deviations of all clusters are given in Table 1. As it can be clearly seen, cluster 1 has least standard deviation and the ranges of \( x \) corresponding to this cluster corresponds to the region of robust design. Figure 6 (b) shows \( \mu \) vs \( \sigma_1 \) plot (\( \mu \) is the mean of weight corresponding to response). One can refer to this plot and choose the cluster depending on one’s requirement. That is, depending on one’s interest to reduce mean or standard deviation, one can choose the regions. For instance, one would choose cluster 1 if looking for least variation in response and would choose cluster 3 in case of least response value but with an acceptable variation of 0.4.
Next, we consider 20 LHS samples of Aspenberg function with 2D input. Equation of function is given in Eq. 9. Figure 8(a) shows the top view of the Aspenberg function. It can be observed that it has deterministic optima in region where $x \in [-0.5,0], y \in [-0.5,0]$. There exist two robust regions at $x \in [0,0.5], y \in [-0.5,0]$ and $x \in [-0.5,0], y \in [0,0.5]$.

![Figure 7](image1.png)  
**Figure 7.** (a) Clusters numbers and (b) $\mu$ vs $\sigma_i$

![Figure 8](image2.png)  
**Figure 8.** (a) Top View with colormap of Aspenberg Function (b) U-matrix and component planes

$$z = 4(x^2 + y^2)e^{-10(x^2+y^2)} + e^{-30y^2} + e^{-100(x-0.25)^2} + e^{-100(x-0.25)^2}$$  \hspace{1cm} (9)

After training SOM over samples and performing clustering which is shown in Figure 7(a), the computed standard deviations are compared in Table 2. It can be observed that cluster 3 shows the largest deviation and other clusters show lesser deviation.

It can be inferred that clusters 1, 2 and 4 are comparatively robust than cluster 1. This can also be validated from $\sigma_i$ values and top view of function Figure 8(a). The above two exercise shows that the modified SOM algorithm along with clustering technique can be used to identify the robust regions in design space. Figure 7(b) shows $\mu$ vs $\sigma_i$ plot.

Next, we consider 50 LHS samples of Aspenberg function with with 4D input given by Eq. 10. Proposed approach gives 5 clusters as output. Standard deviations of all clusters are given in Table 3. Figure 9 shows the U-matrix and component planes. Figure 10(a) shows cluster numbers. It can be inferred from Table 3 that Cluster 4 is the having least standard deviation. Cluster 3 can be ranked as second robust region. Figure 10(b) shows the $\mu$ vs $\sigma_i$ plot. It can be seen that cluster 4 has higher mean compared to other clusters but least standard deviation. Figure 9 shows the U-matrix and component planes.

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Table 2. Summary of Cluster Statistics

<table>
<thead>
<tr>
<th>Cluster</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$x_{\text{median}}, y_{\text{median}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cluster 1</td>
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<td>0.1744</td>
<td>$[-0.4758, -0.0029], [-0.0365, 0.4880]$</td>
</tr>
<tr>
<td>Cluster 2</td>
<td>0.0989</td>
<td>0.1208</td>
<td>$[-0.0355, -0.4816], [-0.1365, 0.4880]$</td>
</tr>
<tr>
<td>Cluster 3</td>
<td>0.2037</td>
<td>0.2517</td>
<td>$[-0.4758, -0.1339], [-0.4835, -0.0671]$</td>
</tr>
<tr>
<td>Cluster 4</td>
<td>0.1542</td>
<td>0.1840</td>
<td>$[-0.1339, 0.4816], [-0.4835, -0.0671]$</td>
</tr>
</tbody>
</table>

Figure 9. U-matrix and component planes

Figure 10. (a) Cluster numbers and (b) $\mu$ vs $\sigma_i$

$$ z = 4 \sum_{i=1}^{4} x_i^2 + \sum_{i=1}^{4} e^{-x_i^3} + \sum_{i=1}^{4} e^{-10(x_i^2-0.25)^2} $$

Table 3. Summary of Cluster Statistics

<table>
<thead>
<tr>
<th>Cluster</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cluster 1</td>
<td>0.2567</td>
<td>0.4555</td>
</tr>
<tr>
<td>Cluster 2</td>
<td>0.2722</td>
<td>0.3688</td>
</tr>
<tr>
<td>Cluster 3</td>
<td>0.1651</td>
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<td>Cluster 4</td>
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</tr>
<tr>
<td>Cluster 5</td>
<td>0.3903</td>
<td>0.3217</td>
</tr>
</tbody>
</table>

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4. Conclusions

This work discusses about using SOM for response function visualisation in design space. The underlying idea is to identify function characteristics such as peaks, valleys, non-linearity of the design space and correlation between the input variables. A modified SOM algorithm is used to address the issue of aberration in classical SOM. This SOM is further clustered based on the weights aiding in identifying flat regions in design space that correspond to robust designs. Further statistical deviations of weights corresponding to response is calculated and then compared with standard deviations of response values obtained from large number of function evaluation in all clusters as validation. Further \( \mu - \sigma \) plots are provided to understand function variability in design space. This allows the designer to make intelligent design decision such as if one wants minimum value of response or minimum variation in response.

References

VARIATIONAL SENSITIVITY ANALYSIS OF ELASTOPLASTIC STRUCTURES APPLIED TO OPTIMAL SHAPE OF SPECIMENS

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Abstract

Aim of this paper is to improve the shape of specimens for biaxial experiments with respect to optimal stress states, characterized by the stress triaxiality. Gradient based optimization strategies are used to achieve this goal. Thus, it is crucial to know how the stress state changes if the geometric shape of the specimen is varied. Design sensitivity analysis (DSA) of the stress triaxiality is performed using a variational approach based on an enhanced kinematic concept that offers a rigorous separation of structural and physical quantities. In the present case of elastoplastic material behavior, the deformation history has to be taken into account for the structural analysis as well as for the determination of response sensitivities. The presented method is flexible in terms of the choice of design variables. In a first step, the approach is used to identify material parameters, thus material parameters are chosen as design variables. Subsequently, the design parameters are chosen as geometric quantities so as to optimize the specimen shape with the aim to obtain a preferably homogeneous stress triaxiality distribution in the relevant cross section of the specimen.

Key words: Specimen Optimization, Shape Optimization, Variational Sensitivity Analysis

1. Introduction

In biaxial tests the stress state can be characterized by the stress triaxiality defined by

\[ \eta = \frac{\sigma_m}{\sigma_{\alpha}} = \frac{I_1(\sigma)}{3 \sqrt{3} J_2(\text{dev} \sigma)} \]  

which classifies the stress state to be tensile if \( \eta > 0 \), compressive if \( \eta < 0 \) or pure shear if \( \eta = 0 \). New specimens for biaxial tests have been developed in [5], which widen the range of stress triaxiality compared to traditional specimens. This is crucial to distinguish between different micro-mechanical damage mechanisms that are responsible for material failure. With the new specimens it is possible to capture various damage mechanisms with only one specimen by varying the loading conditions. These specimens shall be used to identify material parameters for complex damage models. Thus, it is important to be able to separate different damage mechanisms. In this work the optimization goal is to further improve the shape of these specimens so as to achieve a preferably homogeneous stress triaxiality distribution in the area the specimen is predicted to fail. With this at hand, parameter identification for special damage models that are to capture different damage mechanisms will be easier and more accurate.

For design optimization gradient based methods are utilized. The gradient information is determined by means of a variational approach proposed in [1]. It is based on an enhanced kinematic concept that offers a rigorous separation of structural and physical quantities and provides exact gradient information efficiently with moderate effort. Additionally, it allows simultaneous computation of stress states and sensitivities within a finite element framework, c.f. [2]. In contrast to a purely elastic mechanical response, here the deformation history deserves special attention in structural as well as sensitivity analysis, c.f. e.g. [6].

After some preliminaries in section 2 about the viewpoint of kinematics and the notation used in this paper, the model equations of the chosen \( J_2 \) elastoplastic model are summarized in section 3 and the solution strategy is described. Based on this, variational sensitivity analysis is consequently performed and the concept to compute the mechanical response and stress sensitivities are outlined. In section 4 the method is applied to two optimization problems, a parameter identification in section 4.1 and the shape optimization of the chosen X02-specimen in section 4.2. section 5 summarizes the paper and draws a conclusion on the findings.
2. Preliminaries

2.1 Enhanced Kinematics

In the context of structural optimization, the material body is not considered with a fixed reference configuration. Thus, within a general continuum mechanical framework, it is convenient to work with an enhanced kinematic concept. Motivated by an improved viewpoint on the material body using arguments from differential geometry, c.f. [1,3,8], the main idea is the rigorous separation of physical and structural quantities. By the introduction of a fixed parameter space \( B \) with Cartesian basis \( Z \) and local coordinates \( \Theta \), the classical deformation mapping \( \varphi \), defined as

\[
\varphi:\{X,t\} \rightarrow \hat{X}(X,t)
\]  

(2)
can be decomposed into two independent mappings, i.e. the design dependent geometry mapping \( \kappa(\Theta,s) \) and the time dependent motion mapping \( \mu(\Theta,t) \), defined as

\[
\kappa:\{\Theta,s\} \rightarrow X(\Theta,s), \quad \mu:\{\Theta,t\} \rightarrow x(\Theta,t)
\]  

(3)

With the corresponding tangent mappings

\[
K=\nabla_{\Theta} \kappa, \quad M=\nabla_{\Theta} \mu
\]  

(4)

not only the deformation mapping but also the deformation gradient can be decomposed and written as

\[
\varphi=\mu^* \kappa^{-1}, \quad F=MK^{-1}
\]  

(5)

The main advantage of this enhanced viewpoint of kinematics is that implicit dependencies do not arise until the definition of global equilibrium. Note that the parameter \( s \) is used here as a scalar design variable that parameterizes the material body in the reference configuration \( K=K(s) \) and the material points \( X=X(s) \).

2.2 Variations and Derivatives

The total variation of any quantity \( \mathcal{L}(\cdot;u,s;h) \) is given by the sum of the partial derivatives

\[
\delta \mathcal{L}(\cdot;u,s;h)=\delta_u \mathcal{L}(\cdot;u,s;h)+\delta_s \mathcal{L}(\cdot;u,s;h)+\delta_h \mathcal{L}(\cdot;u,s;h)
\]  

(6)

where e.g. the partial variation with respect to \( u \) with fixed \( s \) and \( h \) is defined as

\[
\delta_u(\cdot)=\frac{d}{dz}(\cdot)(s;h;u+\varepsilon \delta u)|_{z=s}.
\]  

(7)

This notation will be used throughout the whole paper.

3. Response and Sensitivity Analysis

Following, the material model, solution strategies to compute the structural response and response sensitivities are outlined. The chosen mechanical model based on a multiplicative split of the deformation gradient \( F=F^p \) can be found in the relevant literature, e.g. [4,9-10] and captures large deformations and strains.

3.1 Mechanical Response

The nonlinear weak form of equilibrium in the actual configuration \( d \) is solved utilizing Newton’s scheme within a finite element framework and reads

\[
r(u,v) = \int_{\Omega} \sigma(u) : \text{grad} \ v + \int_{\Omega} \mathbf{b} \cdot \text{div} \ v - \int_{\Omega} \mathbf{t} \cdot \text{vda}.
\]  

(8)

Based on the strain energy function, given by

\[
W=U(J)+\overline{W}(\overline{\mathbf{F}})=\frac{1}{2} \kappa \left[ \frac{1}{2} (J'-1)- \log J' \right] + \frac{1}{2} \mu |\mathbf{F}-3|
\]  

(9)

stresses can be determined via a Return-Mapping Algorithm, c.f. [4,10,]. Here, \( \overline{\mathbf{F}}=J^{-\frac{1}{2}} \mathbf{F} F^p \) denotes the isochoric part of the elastic contribution of the left Cauchy-Green deformation tensor. The Von Mises type yielding condition only depends on the deviatoric part of the stresses and thus is pressure independent

\[
f=\text{dev} \ \tau-\sqrt{\frac{2}{3} k(a) \leq 0}
\]  

(10)

Hardening during plastic flow is defined by the nonlinear function that depends on the isochoric hardening variable

\[
k(a)=a_o + a_\infty \left[ 1- \exp(-da) \right] + Ha
\]  

(11)

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the material parameters $\sigma_0$ and $\sigma_d$ denote the initial and limit yield stress, respectively. $H$ is the linear hardening modulus and $d$ is a dimensionless parameter. Note that although the deformation process is assumed to be quasi-static, a pseudo-time has to be introduced in order to solve the evolution of the internal hardening variables. The set of history variables that has to be saved in each load step and integration point is $h = \{ F, \overline{b}, \alpha \}$. The evolution of the internal history variables is described by the flow rule

$$L \dot{b} = -\frac{2}{3} \gamma b \cdot \dot{\alpha} = \frac{\sqrt{\gamma}}{\sqrt{3} \gamma}$$  \hspace{1cm} (12)

Due to the nonlinear hardening law, within an implicit time integration, the increment of the plastic multiplier $\Delta \gamma$ has to be solved within a local Newton-Raphson scheme. The solution has to be a Kuhn-Tucker point defined by the conditions

$$\Delta \gamma > 0, \quad f(\tau, \alpha) \leq 0, \quad \Delta f = 0$$  \hspace{1cm} (13)

With the increment of the plastic multiplier at hand, the stress tensor can be determined

$$\text{dev } \tau = \text{dev } \tau^p = 2\mu \Delta m$$  \hspace{1cm} (14)

with $\mu = \frac{1}{3} \tau \overline{b}$ and the flow direction $n = \frac{\text{dev } \tau^p}{\| \text{dev } \tau^p \|}$. Due to the assumption of incompressible plastic flow the hydrostatic pressure can be easily computed to

$$p = p^p = U'(J)$$  \hspace{1cm} (15)

Note that the total Kirchhoff stress tensor reads $\tau = J \mu I + \text{dev } \tau$ and the Cauchy stress tensor is given by $\sigma = J^{-1} \tau$. Consistent linearization of the stress tensor has been derived in [9-10] and the numerical solution strategy is outlined. An excellent summary of the solution algorithm can also be found in [4] in the context of isogeometric analysis.

3.2 Response Sensitivity

Response sensitivities are obtained variationally at continuous level. As Eq. (8) has to hold for any design change $\delta \sigma$, its total variation has to vanish, i.e., \[ \delta \sigma = \delta \sigma_x + \overline{\delta \sigma} + \delta \sigma_y = 0. \]  \hspace{1cm} (16)

The third addend corresponds to the deformation history that has to be captured and considered for all variations, i.e., \[ \delta \sigma_x = \delta \sigma_x = \delta \sigma_y = 0. \]  \hspace{1cm} (16)

From the matrix form of Eq. (16),

$$\delta \mathbf{R} = \left[ \frac{\partial \mathbf{R}}{\partial \mathbf{u}} \right] \delta \mathbf{u} + \left[ \frac{\partial \mathbf{R}}{\partial \mathbf{s}} \right] \delta \mathbf{s} + \left[ \frac{\partial \mathbf{R}}{\partial \delta \mathbf{h}} \right] \delta \mathbf{h} = \mathbf{K} \delta \mathbf{u} + \mathbf{P} \delta \mathbf{s} + \mathbf{H} \delta \mathbf{h}. $$  \hspace{1cm} (17)

the sensitivity matrix $\mathbf{s}$ can be derived, that is the total derivative of the structural response $\mathbf{u}$ w. r. t. design changes

$$\delta \mathbf{u} = - \mathbf{K}^{-1} \left[ \mathbf{P} \delta \mathbf{s} + \mathbf{H} \delta \mathbf{h} \right] = - \mathbf{K}^{-1} \left[ \mathbf{P} + \mathbf{H} \mathbf{G} \right] \delta \mathbf{s} = \mathbf{S} \delta \mathbf{s} $$  \hspace{1cm} (18)

where the matrix $\mathbf{G}$ denotes the total design derivative of the history variables

$$\delta \mathbf{h} = \left[ \frac{\partial \mathbf{h}}{\partial \mathbf{s}} \right] \delta \mathbf{s} + \left[ \frac{\partial \mathbf{h}}{\partial \delta \mathbf{h}} \right] \delta \delta \mathbf{h} = \mathbf{G} \delta \mathbf{s} $$  \hspace{1cm} (19)

Here, $\mathbf{K}$ denotes the tangent stiffness matrix. $\mathbf{P}$ denotes the so-called pseudo-load matrix and $\mathbf{H}$ is the history sensitivity matrix. Note that assuming $\mathbf{h}_0 = 0$ and $\mathbf{h}_0 = 0$ at time $t_0$, from Eq. (18) we obtain

$$\mathbf{S} = - \mathbf{K}^{-1} \mathbf{P} $$  \hspace{1cm} (20)

for the first pseudo-time increment. The sensitivity part corresponding to the deformation history has to be evaluated at the end of each time step that causes plastic yielding and saved for the subsequent step.

3.3 Stress Sensitivity

The gradient of the stress triaxiality as defined in Eq. (1) can be computed following the same concepts, which leads to

$$\delta \eta = \delta \eta \delta \sigma + \delta \eta \delta \sigma $$  \hspace{1cm} (21)

where the total variations of the mean and Von Mises equivalent stress read

$$\delta \sigma = \delta \sigma \delta \sigma + \delta \sigma \delta \sigma $$  \hspace{1cm} (22)

Thus, it is essential to calculate the total variation of the Cauchy stress tensor $\sigma$ given by

$$\delta \sigma = \delta \sigma \sigma + \delta \sigma \sigma + \delta \sigma \sigma $$  \hspace{1cm} (23)

which follows the same principles as the variation of the equilibrium condition, i.e., Eq. (16), and is straightforward.

\[ \text{145} \]
4. Optimization Tasks

4.1 Parameter Identification

In a first step the method mentioned above is used to fit the material parameters of the constitutive model to the material behavior by means of a standard tensile test, c.f. Figure 1. In this case the design variables are chosen as the material parameters $s = m = [E \ v \ \sigma_0 \ \sigma_* \ \sigma_{\text{f}} \ H \ d]^T$.

The initial values of the material parameters are taken from [5] and summarized below:

$$E = 69 \ \text{GPa}, \quad v = 0.3, \quad \sigma_0 = 320 \ \text{MPa}, \quad \sigma_* = 100 \ \text{MPa}, \quad H = 650 \ \text{MPa}, \quad d = 30.$$

The geometry is discretized with 1539 hexahedral $F$-elements that are free of volumetric locking, c.f. e.g. [4,7]. In all the mechanical problem counts 4872 degrees of freedom. Due to symmetry only one eighth of the geometry is modeled. Thus, the measured reaction force is divided by 4 for the comparison. Symmetric boundary conditions are applied and a maximum displacement of 10 mm is prescribed within 200 linear steps, that is the point where the maximum reaction force has been measured. The time discretization implies that the number of data points that are compared with the simulation is $\text{ndp} = 200$.

![Figure 1. Tensile test (left) and FE model (right)](image)

The optimization problem is to minimize the error between the load-displacement curves of the experiment $F$ and the simulation $f(m)$ depending on the set of material parameters. We can define the optimization problem as

$$\min_{s, t} \left \| f(m) - F \right \|_2^2, \quad s, \ t, \ m' \leq m \leq m'_u. \quad (24)$$

The problem is solved utilizing the MATLAB solver lsqnonlin.

![Figure 2. Optimization process: Load-displacement curves (left) and Objective history (right)](image)

Figure 2 summarizes the optimization process. 22 iterations are needed to finally reach the stopping criterion, that is the relative change of the objective compared to the prior iteration is less than $1 \times 10^{-6}$. The parameters are identified to

$$E = 61.92 \ \text{GPa}, \quad v = 0.31, \quad \sigma_0 = 313.3 \ \text{MPa}, \quad \sigma_* = 119.2 \ \text{MPa}, \quad H = 548.7 \ \text{MPa}, \quad d = 10.72.$$

With this at hand, the next optimization task, that is the shape optimization of a chosen specimen, can be conducted.

4.2 Specimen Shape Optimization

The chosen, so-called X02-specimen, is shown in Figure 3 together with the FE model and the geometry constraints. Due to symmetry only one eighth of the geometry has to be discretized. 5928 hexahedral $F$ elements are used and the mechanical problem counts 23,535 degrees of freedom. Within 200 linear steps the maximum displacement of 0.75 mm is applied to all sides of the specimen. As the shape of the specimen is to be optimized, geometric parameters are chosen as design variables $s = X = [R_i, R_o, R_t, d]$, namely.

$R_i$: inner radius, $R_o$: outer radius, $R_t$: radius in thickness direction, $d$: penetration depth.
Aim of the shape optimization is to achieve a preferably homogeneous stress triaxiality in the cross section. For this, the length of the vector of stress triaxiality in the cross section is maximized. Additionally, during the optimization the cross section has to stay constant, which is defined by the equality constraint

\[ c^o = t_e \cdot l_e - 12 \text{ mm}^2 = 0 \]  

(25)

c.f. Figure 3. To prevent the appearance of sharp corners at the edges of the notch in thickness direction the penetration depth should not be larger than the radius in thickness direction. This is captured with the inequality constraint

\[ c^n = d - R_i \leq 0 \]  

(26)

Consequently, the optimization problem is defined as

\[ \max_X J(X) = ||\eta(X)||^2 \]  

s. t. \[ c^o = 0, \]

\[ c^n = 0, \]

\[ X^i \leq X^e \leq X^u. \]

The problem is solved utilizing the MATLAB solver \texttt{fmincon}.

Figure 4 summarizes the result of the optimization process. After 28 iterations the solver found a solution. Due to the equality constraint, the maximum stress triaxiality does not change significantly, but the distribution over the cross section is much more homogeneous in the optimized case than it was for the initial shape. The initial and optimized values of the design parameters are summarized in Table 1. The values of the equality and inequality constraints at converged state are

\[ c^o = 3.2925e-10, \quad c^n = -8.9378e-4. \]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial value [mm]</th>
<th>Optimized value [mm]</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_1 )</td>
<td>3.003</td>
<td>1.889</td>
<td>19.314</td>
</tr>
<tr>
<td>( R_2 )</td>
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<td>1.570</td>
<td></td>
</tr>
<tr>
<td>( R_3 )</td>
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<td>1.272</td>
<td></td>
</tr>
<tr>
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5. Summary and Conclusions

In this work an efficient method for the computation of design sensitivities for structures with elastoplastic material behavior has been presented. The method is based on a variational approach that allows simultaneous computation of structural response and response sensitivities using the direct differentiation method. The obtained gradient information can be used to feed gradient based optimization algorithms.

The method has been applied to two different optimization problems. First, a parameter identification has been performed to fit the model parameters to the real material (AlCuMgSi). The curve fitting procedure resulted in an excellent agreement with the experimentally measured load-displacement curve.

In a second example the shape of the chosen X02-specimen has been optimized with the aim to achieve a preferably homogeneous stress triaxiality distribution. The stress triaxiality characterizes the stress state of the parts of the structure and can be used to classify different damage mechanisms. In the presented example the length of the vector of stress triaxiality in the relevant part of the specimen has been maximized. The optimization resulted in a slightly different shape, which shows a much more homogeneous stress triaxiality distribution in the relevant part of the specimen.

Future investigations will address the validation of the results in terms of manufacturing the optimized shapes and test them. Further, different specimen types and also different loading scenarios will be analyzed.

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STRAIN-ACTUATED SOLAR ARRAYS FOR SPACECRAFT ATTITUDE CONTROL ASSISTED BY VISCOELASTIC DAMPING

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Abstract

This article presents a utilization of viscoelastic damping to reduce control system complexity for strain-actuated solar array (SASA) based spacecraft attitude control systems (ACSs). SASA utilizes intelligent structures for attitude control, and is a promising next-generation spacecraft ACS technology with the potential to achieve unprecedented levels of pointing accuracy and jitter reduction during key scientific observation periods. The current state-of-the-art SASA implementation utilizes piecewise modeling of distributed piezoelectric (PZT) actuators, resulting in a monolithic structure with the potential for enhanced ACS reliability. PZT actuators can operate at high frequencies, which enables active vibration damping to achieve ultra-quiet operation for sensitive instruments. Relying on active damping alone, however, requires significant control system complexity, which has so far limited adoption of intelligent structures in spacecraft control systems. Here we seek to understand how to modify passive system design in strategic ways to reduce control system complexity while maintaining high performance. An integrated physical and control system design (co-design) optimization strategy is employed to ensure system-optimal performance, and to help understand design coupling between passive physical aspects of design and active control system design. In this study, we present the possibility of utilizing viscoelastic material distributed throughout the SASA substructure to provide tailored passive damping, intending to reduce control system complexity. At this early phase of the study, the effect of temperature variation on material behavior is not considered; the study focuses instead on the design coupling between distributed material and control systems. The spatially-distributed design of both elastic and viscoelastic material in the SASA substructure is considered in an integrated manner. An approximate model is used that balances predictive accuracy and computational efficiency. This model approximates the distributed compliant SASA structure using a series of rigid links connected by generalized torsional springs and dampers. This multi-link pseudo-rigid-body dynamic model (PRBDM) with lumped viscoelastic damping models is derived, and is used in numerical co-design studies to quantify the tradeoffs and benefits of using distributed passive damping to reduce the complexity of SASA control systems.

Key words: Spacecraft Attitude Control System, Strain-Actuated Solar Array, Viscoelastic Damping, Co-Design

1. Introduction

After the success of the Hubble and other space telescopes, goals for next-generation space observatories, such as the James Webb space telescope, demand higher pointing precision [1], enabling discoveries such as finding the first luminous sources and reionization histories from the early universe. The angular orientation of current spacecraft, including state-of-the-art space observatories, is controlled by reaction control system (RCS) thrusters, momentum exchange devices including reaction wheel actuators (RWA), and other devices [1-2]. These conventional attitude control systems (ACS), however, often make unwanted disturbances (jitter) that hinder precise pointing for clear and noise-free deep space observation [3]. Recent studies suggest that small slewing maneuvers, precision pointing, and jitter suppression can be simultaneously achieved by using strain-actuated solar arrays (SASAs), a novel ACS based on intelligent structures [4-5].

The SASA architecture proposed earlier includes piezoelectric (PZT) strain actuators bonded to solar panel surfaces
and/or on the panel-to-panel vane for a small-angle control, and a solar array (SA) root actuator for larger slewing motion [4]. One drawback of the SASA architecture is that the control system for the root actuators and strain actuators needs to handle all vibration frequencies actively via PZTs, which demands a high-frequency, sophisticated control system. A possible remedy is to add strategic passive damping to the SASA system to manage high-frequency vibration modes. Active control can then focus more on low-frequency dynamics, with possibly more aggressive control laws than without tailored passive damping. An initial effort introduced a conceptual viscoelastic damping system to reduce ACS system complexity by passively diminishing vibrations [6]. The model developed for this previous work included a full two-dimensional (2D) spacecraft model with two single-joint solar panels using a single revolute-joint pseudo-rigid-body dynamic model (1R-PRBDM). Improvement in peak vibration amplitude reduction was observed, but adding viscoelastic material did not improve overall system performance (maximum slew angle). This may have been due to omission of material functions (i.e., function-valued material properties) as design variables in the study, and a performance improvement is expected by exploring the optimal material function parameters [7].

In this study, we expand on previous work by investigating SASA with viscoelastic damping involving: (1) a multiple revolute-joint pseudo-rigid-body dynamic model (nR-PRBDM) [8], (2) a unit test model of the SASA system, (3) design of the viscoelastic material function, and (4) more realistic modeling of a revolute-joint damping component using a 2D axisymmetric shear flow model. The design problem we propose in this study contains torque control trajectories at each revolute joint and viscoelastic fluid material function parameters as design variables. A simultaneous material and control optimization formulation are used, based on a specific spacecraft maneuver. Figure 1(a) illustrates the nR-PRBDM SASA modeling concept. Total system momentum must be conserved in this configuration; SASA controls spacecraft attitude via momentum transfer. As a simplification for this study, the rigid-body mode is eliminated, as illustrated in Figure 1(b). This still supports the study of the movement of a multi-body system with active control and viscoelastic passive damping. Reference [6] details the full spacecraft PRBDM shown in Figure 1(a), which is outside the scope of this article.

2. Methods

2.1 Viscoelastic Damper Model

For the 2D axisymmetric model, we assume that the revolute joints shown in Figure 1 are concentric cylinders with viscoelastic fluid in the gap, as shown in Figure 2(a). By setting the frame of reference attached to the outer cylinder, net angular joint motion is defined as the rotation of the inner cylinder relative to the rotation of the outer cylinder. We assume that the outer cylinder is attached to the panel member closer to the spacecraft body, while the inner cylinder is attached to the panel member located farther from the center. Figure 2(b) shows a top view of the concentric cylinders and a uniformly distributed numerical mesh nodes inside the viscoelastic fluid region. For the 1D reduced-order model, we do not compute the fluid shear inside the gap, but assuming a one-dimensional (1D) translational viscoelastic dashpot in the rotational direction, as depicted in Figure 2(c).
2.1.1 Two-Dimensional Axisymmetric Model

The 2D axisymmetric model in a cylindrical coordinate was derived based on the three-dimensional (3D) Cauchy momentum equations [9]. Assuming 2D axisymmetry and linear viscoelasticity, the following assumptions are made: \( u = u_t = 0 \), \( \partial / \partial \theta = \partial / \partial z = 0 \), \( \tau_r = \tau_\theta = \tau_\phi = \tau = 0 \), and \( g_r = g_\theta = g_\phi = 0 \). The original form of the 3D Cauchy momentum equations is given in Eqs. (1)-(3):

\[
\frac{\partial u_r}{\partial t} + \frac{u_r}{r} \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_z}{r} \frac{\partial u_r}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{1}{\rho} \frac{\partial (\tau_{rr})}{\partial r} + \frac{1}{\rho} \frac{\partial \tau_{\theta \theta}}{\partial \theta} + \frac{1}{\rho} \frac{\partial \tau_{zz}}{\partial z} - \frac{\tau_r}{\rho} + g_r
\]

(1)

\[
\frac{\partial u_\theta}{\partial t} + \frac{u_r}{r} \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_z}{r} \frac{\partial u_\theta}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial \theta} + \frac{1}{\rho} \frac{\partial (\tau_{r\theta})}{\partial r} + \frac{1}{\rho} \frac{\partial (\tau_{\theta \theta})}{\partial \theta} + \frac{1}{\rho} \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_\theta}{\rho} + g_\theta
\]

(2)

\[
\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_\theta \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{\rho} \frac{\partial (\tau_{rz})}{\partial r} + \frac{1}{\rho} \frac{\partial (\tau_{\theta z})}{\partial \theta} + \frac{1}{\rho} \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_z}{\rho} + g_z
\]

(3)

The governing equations given in Eqs. (4)-(5) are obtained by applying the above assumptions. This rotational shear flow is driven solely by shear stress acting in the \( \theta \)-direction on surfaces with a normal in the \( r \)-direction and not by the pressure gradient; thus, Eq. (4) represents the resultant \( r \)-direction pressure gradient and does not affect the flow. Consequently, we can use Eq. (5) as the flow governing equation for the 2D axisymmetric model:

\[
\frac{u_z}{r} = \frac{1}{\rho} \frac{\partial p}{\partial \theta}
\]

(4)

\[
\frac{\partial u_\theta}{\partial t} = \frac{1}{\rho} \frac{\partial p}{\partial \theta} + \frac{1}{r^2 \rho} \left( 2 \tau_{\theta \theta} + r^2 \frac{\partial \tau_{\theta \theta}}{\partial \theta} \right).
\]

(5)

The linear viscoelasticity assumption does not involve normal stress behavior. Thus, from the shear stress tensor, the term in the \( \theta \)-direction with respect to the \( r \)-direction only remains non-zero. The constitutive equation for the shear stress term in Eq. (5) is:

\[
\tau_\theta(t) = \tau_\theta(t) = \int_{-\infty}^{t} G(t-t') \dot{\gamma}(t') \, dt'
\]

(6)

and if we assume that the system is stationary and the viscoelastic fluid has fully relaxed before the time horizon of interest, we can eliminate the integration before the time \( t = 0 \), making the interval of the integration \( (0,t) \) instead of \((-\infty,t)\). Also, the stress relaxation kernel \( G(t-t') \) diminishes very quickly, being negligible after some finite time \( (t-t') > t^* \). That is, the integral is approximated as zero for \( t' < (t-t^*) \), giving the approximation:

\[
\int_{-\infty}^{t} G(t-t') \dot{\gamma}(t') \, dt' \approx \int_{t^*}^{t} G(t-t') \dot{\gamma}(t') \, dt'
\]

(7)

where \( t^* \) is determined by the smallest value that satisfies \( G(t^*) \approx 0 \) with a small tolerance. We use the single-mode Maxwell model for the kernel function, \( G(t) = G_\epsilon \exp(-t/\lambda_\epsilon) \), for simplicity. The relevant shear rate component in the cylindrical coordinate system is defined as:

\[
\dot{\gamma}_\theta = \dot{\gamma}_\theta = \frac{\partial u_\theta}{\partial r} - \frac{u_{\theta \theta}}{r}
\]

Where \( u_{\theta \theta} \) is defined within \( R \leq r \leq aR \), \( a \) is the ratio of the radii of the outer to the inner cylinders, \( R \) is the radius of the inner cylinder, and values of these parameters are \( a = 1, 25 \) and \( R = 0, 008 \) m.

2.1.2 One-Dimensional Reduced-Order Model

The reduced-order model with a 1D translational viscoelastic dashpot in the rotational direction (Figure 1(c)) does not require a shear flow computation. The motion of the revolute joint can be approximated using a translational movement at the radius of middle-plane between the inner and outer cylinders. The constitutive equation for the shear stress is used directly for computing torque in the SASA dynamic model:

\[
\tau_\theta(t) = \tau_\theta(t) = \int_{-\infty}^{t} G(t-t') \left[ a + \frac{1}{2} R H (t-t') \right] dt' \approx \frac{a + 1}{2} R \int_{-\infty}^{t} G(t-t') \dot{\theta}(t') \, dt' + \int_{t^*}^{t} G(t-t') \dot{\theta}(t') \, dt'
\]

(9)

The shear stress computed using Eqs. (6) or (9) is used directly in the viscoelastic torque input \( T_\theta = \alpha_\tau \tau_\theta \) for system dynamics prediction, where \( \alpha_\tau \) is a lumped parameter that carries the geometric information of the joint, defined as:

\[
\alpha_\tau = 2\pi (aR)^2 H = 1, 2566 \times 10^{-3} \text{ m}^3
\]

(10)

where \( H \) is the height of the joint geometry, and the values of these parameters are set to: \( a = 1, 25 \), \( R = 0, 008 \) m, and \( H = 0, 2 \) m. Similarly, the relationship between angular displacement (or angular velocity) and shear (or shear rate), \( \theta = a_\theta \dot{\gamma}, \dot{\theta} = a_\theta \dot{\gamma} \), can be described using a lumped parameter \( a_\theta \), which is defined as:

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\[ \omega = \frac{2}{R(a + 1)} = 1.111 \times 10^3, \]  

(11)

2.2 Multiple Revolute-Joint Pseudo-Rigid-Body Dynamic Model (nR-PRBDM)

This model (developed and presented in Ref. [8]) was derived to approximate compliant mechanism dynamics for use in design optimization studies, and validated for several beam configurations using finite element results. Here this model is used for design studies with both active torques and those exerted by the passive viscoelastic damper.

2.3 Design Problem Formulation

The control problem is defined within a finite time horizon with an initial position of the SASA panel and a specified final target position and velocity. The design optimization problem considered here is given in Eq. (12):

\[
\begin{align*}
\text{minimize} & \quad f = \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} T_i \dot{\theta}_i^2 \, dt \\
\text{subject to} & \quad (0, 9\dot{\theta}_r - \dot{\theta}_r) = 0, (0, 5\dot{\theta}_r - \dot{\theta}_r) = 0, (0, 4\dot{\theta}_r - \dot{\theta}_r) = 0, \quad T_r - T_r = 0 \quad /\text{BCs}
\end{align*}
\]

where

\[
\begin{align*}
\dot{\theta}_i &= \begin{bmatrix} \dot{\theta}_1 \cr \dot{\theta}_2 \cr \vdots \cr \dot{\theta}_n \end{bmatrix} \quad /\text{states ICs} \\
\dot{\theta}_i &= \begin{bmatrix} \dot{\theta}_1 \cr \dot{\theta}_2 \cr \vdots \cr \dot{\theta}_n \end{bmatrix} + (\ddot{\theta}_i - \ddot{\theta}_r) \quad /\text{time-marching}
\end{align*}
\]

\[
M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + B\dot{\theta} + K\theta - T_r(\tau_0, \tau_0) - T_r = 0 \quad /\text{dynamics}
\]

(12)

where a symbol with an overhead arrow represents a vector of discrete time points (e.g., \( \dot{T}_i = [T_1, T_2, \ldots, T_n] \)), and a symbol with an arrow below is a vector of n-joints (e.g., \( \dot{\theta} = [\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n] \)). The objective function is a sum of integrals of squared control torque, which is a proxy of control effort from an energy perspective. Other possible objectives include maximizing slew angles in a given time period, or minimizing time required to achieve specific slew motion. The equality constraints define the boundary conditions on position and velocity at the final time, \( t_f \). Future modeling may also include acceleration at the final time as another constraint to avoid further oscillation of the system after the final time. Initial conditions at the time \( t_0 \) are specified. The Euler forward method is used for integration within a single-shooting solution strategy. The dynamics equation is provided in a second-order differential equation form with a torque term exerted by the viscoelastic damper, \( T_r \), and another torque term that represents the active torque control input, \( T_a \). Note that \( M \) is the mass matrix at a certain angular pose \( \theta \); \( C \) is the Coriolis effect matrix; \( B \) is the frictional loss term (zero here); \( K \) is the stiffness matrix that quantifies passive linear torsional springs at each revolute joint.

3. Results

Figure 3(a) shows the optimal material functions for the 2D axisymmetric and 1D reduced-order model cases with a maximum control torque constraint of 10 Nm. The dotted line denotes stress relaxation modulus and the red square denotes Maxwell parameters of optimal material functions for the 2D axisymmetric case. The solid line and the black circle correspond to those material characteristics for the 1D reduced-order model case. Figure 3(b) shows the corresponding optimal control torque trajectories. Control torque trajectories are represented using a piecewise cubic Hermite interpolating polynomial with 20 equally-distributed mesh nodes.

3.1 Two-Dimensional Axisymmetric Model Design Results

Optimal Maxwell model parameters are \( G_r = 0.943 \) MPa and \( \lambda_r = 1.36 \times 10^{-3} \) s. Displacement trajectories shown in Figure 4 are smooth and satisfy the final time boundary condition with a maximum constraint violation of 6.486 \( \times \) \( 10^{-3} \). Total computational time was 242 minutes for 3,186 function evaluations with a dual 2.1GHz Xeon Gold 6130.
(32-core, 64-thread) workstation with the MATLAB interior-point nonlinear programming algorithm. We anticipate these results to be close to planned experimental data due to the fairly realistic model. The optimal control design exhibits some higher-frequency oscillations to mitigate vibrations due to viscoelastic material memory effects. Although viscoelastic materials can absorb a wide-range of vibration frequencies, a comprehensive evaluation of their value for SASA is not yet complete. A rigorous comparison between viscous and viscoelastic dampers will be required for determining the right choice for this type of system. In addition, more flexible viscoelastic models may support broader design exploration and enhanced system performance.

3.2 One-Dimensional Reduced-Order Model Design Results

The Maxwell model parameters are $G^* = 1.09$ MPa and $\lambda^* = 8.69 \times 10^{-4}$ s, corresponding to larger initial stress and smaller relaxation time compared to the 2D axisymmetric model results. Control trajectories do resemble those from the 2D model study, but with less pronounced oscillations. Displacement trajectories satisfy the final time boundary condition with a maximum constraint violation of $1.033 \times 10^{-10}$. Total computational time was 13 minutes for 8,375 function evaluations (same hardware as before). This reduction in computational expense would enable more extensive design studies, such as a longer time horizons for multi-stage maneuvers. While control trajectories are sufficiently close to informing later design activities, the difference between Maxwell model parameters is not small, corresponding to distinct materials. One possible reason for this gap is that the 2D model accounts for transient shear flow behavior (inertia of the viscoelastic fluid), and the 1D model does not. Another possible source of disparity is the conversion of the rotational shear movement to simple translational movement. This simplification is not physically rigorous, and may require additional tuning parameters to simulate the complex shear flow behavior without solving actual shear flow mechanics.

Figure 3. Optimal designs for the 2D axisymmetric and 1D reduced-order models (maximum control torque; 10 N-m)

(a) Material function design  
(b) Corresponding control torque design

Figure 4. Dynamics of the system resulting from optimal design based on the 2D axisymmetric model

(a) Angular displacement trajectories  
(b) Angular velocity trajectories

3.3 Discussions on Limiting Maximum Amplitude of the Active Control System

Figure 5(a) shows the optimal material functions for the 2D axisymmetric and 1D reduced-order model cases with varying maximum control torque constraints from 8 to 20 N-m, and Figure 5(b) shows the corresponding optimal control torque trajectories. The case with the 20 N-m constraint does not limit the control torque, and cases with a
constraint of 11 N-m and above limit the active control system very loosely. However, maximum control torque values of 10, 9, and 8 N-m cases influence the control trajectory significantly. From the material functions plot, we observe that optimal fluid design moves to higher stress relaxation time and amplitude when the control torque design space is significantly limited. Also, we observe from the trend of the optimal control torque trajectories that active control burden is distributed to multiple revolute joints when the maximum capability of active control amplitude is limited. As we see from the blue colored lines in Figure 5(b), limited control at one revolute joint (RJ) can be compensated for by having more aggressive control at the other RJs.

![Material function design](image1)

(a) Material function design

![Corresponding control torque design](image2)

(b) Corresponding control torque design

Figure 5. Optimal designs based on the 1D reduced-order models with varying maximum control torque constraints

3. 4 Discussions on Vibration Frequencies

The period of the natural frequencies of the system ranges from 1.78 to 30.3 s (order of $10^1$ to $10^2$ s) and the control torque input is exciting the system with periods on the order of $10^{-2}$ s. These are at least one or more orders of magnitude larger than the characteristic relaxation time of the optimal viscoelastic fluid ($\sim 10^{-3}$ s). The optimal viscoelasticity is clearly that of a fluid with a short relaxation time compared to dominant timescales in the system dynamics, though the specific relationships between these timescales are not yet clear. Our future work with purely Newtonian fluids and viscoelastic fluids with different relaxation time ranges will study this in more depth.

4. Conclusion

The use of viscoelastic damping in intelligent structures, such as SASA, was studied here from an integrated design optimization perspective. These studies were performed using both a higher fidelity model (2D axisymmetric, transient shear flow) and a simplified model (1D reduced-order model). The latter significantly reduces computational cost, and results show similar system dynamics. Additional tuning parameters, however, may be needed to improve optimal material property identification. Viscoelasticity may help mitigate a wide range of vibration frequencies, but design effort may be costly. Future work will investigate the potential benefits of incorporating viscoelastic elements into SASA systems, and whether the additional design effort is worth the performance improvement.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. CMMI-1653118.

References


STRUCTURAL OPTIMIZATION OF ELASTIC-PLASTIC CONTACT PROBLEMS USING THE LEVEL SET METHOD

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Abstract

This work aims to analyze and to solve numerically the shape and topology optimization problem for two bodies in bilateral frictional contact. The static elasto-plastic material model with linear kinematic hardening rather than elastic material model is assumed. This contact phenomenon is governed by the system of the coupled variational inequalities. The structural optimization problem consists in finding such shape of the domain occupied by the body in contact or material distribution to minimize the contact stress. Using regularization and penalization techniques the original system of variational inequalities is transformed into the system of nonlinear equations. The shape derivative of the cost functional is calculated using the material derivative method. In the numerical computations the level set method is used to describe the evolution of the domain shape in the design space. Generalized Newton method combined with gradient method is used to solve numerically this problem. The results of computation are provided and discussed.

Key words: Structural Optimization, Contact Problems, Elastic-Plastic Materials, Level Set Method

1. Introduction

Topology optimization consists of material distribution within the design domain to minimize the given cost functional describing required features of the structure. Most research related to topology optimization has been concerned with linear elastic structures (see references in [11]). The amount of papers dealing with non-linear elastic structures or nonlinear mechanical behavior is limited. The ability to take elasto-plastic materials into account is of great interest of industrial engineers.

The contact problems between elasto-plastic bodies are discussed in [4]. Since the stresses in contact area are very high and result in yielding of the material the modeling of contact phenomenon should include both elastic and plastic deformation. Paper [4] reviews contact modelling approaches for different geometries and types of contact. Optimal control of contact problems for elasto-plastic materials have been considered in [1, 16]. Paper [1] deals with optimal control for quasi-static bilateral frictional elastic contact problem with tracking type cost functional and the boundary traction or the volume force as the design variables. The necessary optimality conditions are formulated using the regularization and penalization techniques. In [16] hot rolling process is modeled by dynamic frictional large displacement elasto-plastic contact problem. An optimal control problem to achieve a desired shape of a work piece is formulated and analyzed. A numerical algorithm is proposed and validated numerically to solve hot rolling problems.

Structural optimization of elasto-plastic structures has been investigated in [7, 9, 10, 15]. A major difficulty in shape or topology optimization of elasto-plastic problems as the path-dependent problems is the sensitivity analysis [9, 15]. For linear elastic problems the sensitivity analysis causes limited difficulties and it is possible to formulate the convex separable approximations. However, for path-dependent problems the situation is more complicated. Due to the material path-dependence, the sensitivity will also be path-dependent and as a consequence the sensitivity will be obtained by the solution to a terminal value problem. Therefore the original structural optimization problem has to be regularized either using the filter technique or different penalization techniques.

The aim of this work is to analyze and to solve numerically the shape and topology optimization problem for two bodies in contact assuming static elasto-plastic rather than elastic material model with linear kinematic hardening. The optimization problem consists in finding such shape of the domain occupied by the body in contact to minimize the con-
act stress. In the paper penalization approach will be used to regularize the structural optimization problem. Moreover, the level set approach [13] will be used to solve it numerically. The small strain plasticity material model is used [6]. The optimization problem is discretized using the finite element. The semi-smooth Newton method and the descent gradient method are used to solve the discrete contact problem and the optimization problem respectively. Numerical results are provided and discussed.

2. Contact Problem

Consider a body occupying a bounded domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, with a Lipschitz continuous boundary $\Gamma$. The boundary $\Gamma$ is divided into three open disjoint and measurable parts $\Gamma_1, \Gamma_2$, and $\Gamma_3$ such that $\text{meas} (\Gamma_1) > 0$. The body is loaded by a given volume force of density $f_1 = f_1(x)$ in $\Omega$ and a given surface traction of density $f_2 = f_2(x)$ applied on the boundary $\Gamma_1$. The body is clamped along the boundary $\Gamma_1$, i.e., its displacement $u = (u, u_c), u = u(x), x \in \Gamma_1$, vanishes there. Along the boundary $\Gamma_3$ the body is assumed to be in bilateral frictional contact with the rigid foundation. The friction phenomenon is modeled by the Coulomb’s law. The forces $f_1$ and $f_2$ are acting slow enough to neglect the inertial terms. The domain $\Omega$ is filled with a material undergoing under the loading of volume or boundary forces elasto-plastic deformation including kinematic and isotropic hardening. In the elastic range it obeys Hooke’s law [1, 6] defined by a fourth-order tensor $C$ such that, for any symmetric matrix $\xi$, $C\xi = 2\lambda \xi + \mu tr(\xi)I_d$, where $\lambda$ and $\mu$ are the Lamé constants and $I_d$ and $tr$ are identity and trace operators, respectively. Let us also introduce the Cauchy stress tensor $\sigma$ and the linearized strain tensor $\varepsilon$ [1, 6], respectively

$$\sigma = \sigma(u) = [\sigma_{ij}]$$ and $$\varepsilon = \varepsilon(u) = \frac{1}{2}(u_{ij} + u_{ji}), i, j = 1, \ldots, d, u_{ij} = \frac{\partial u_i}{\partial x_j}.
$$

(1)

The summation convention over repeated indices [1, 6] is used throughout the paper. The divergence operator $\text{div}(\sigma)$ of a second order tensor $\sigma$ is defined as $\text{div}(\sigma) = \frac{\partial \sigma_{ij}}{\partial x_i}, i, j = 1, \ldots, d$. We assume the plastic deformation of the material is governed by the additive small strain plasticity model [4]. In this model the material strain $\varepsilon$ is sum of the elastic strain $\varepsilon'$ and the plastic strain $\varepsilon^p$, i.e.,

$$\varepsilon(u) = \varepsilon'(u) + \varepsilon^p(u)
$$

(2)

and the stress tensor satisfies

$$\sigma(u) = C\varepsilon'(u) = C(\varepsilon(u) - \varepsilon'(u)) \quad \text{and} \quad \varepsilon(u) = C^{-1}\sigma(u) + \varepsilon'(u).
$$

(3)

The plastic deformation with the hardening phenomenon is governed by the generalized plastic strain $(\varepsilon^p, \xi)$ and the generalized plastic stresses $(\sigma, \chi)$. The variable $\xi$ represents a set of scalar or tensorial internal variables capturing the feature of hardening behavior [6]. The back stress variable $\chi$ denotes an internal force arising during the hardening and causing the translation of the initial yield surface. The back stress $\chi$ and the internal variable $\xi$ are related by [6]

$$\chi = -H\xi \text{ in } \Omega
$$

(4)

where $H$ denotes the hardening tensor. The generalized plastic stress may take values only in a closed convex set $K$ of admissible generalized stresses. For a given yield function $\phi$ this set is defined as [6]

$$K = \{\sigma, \chi : \phi(\sigma, \chi) = 0\},
$$

(5)

The elastic region of the loaded material corresponds to $\phi(\sigma, \chi) < 0$, i.e., the interior of the cone $K$. In this region the plastic strain $\varepsilon'(u)$ is equal to zero, i.e., $\varepsilon'(u) = 0$. On the boundary of the cone $K$ called the yield surface, where $\phi(\sigma, \chi) = 0$, the plastic strain can be nonzero. The evolution of the plastic strain $\varepsilon'$ and the internal variable $\xi$ is governed by the associative flow rule [6, 9] stating that the rates of plastic strain and the internal variable belong to the normal cone $N_\phi$ to the set $K$ at a point $(\sigma, \chi)$. For the smooth yield function, it implies the existence of the nonnegative scalar $\lambda'$ and the Kuhn–Tucker complementarity conditions relating $\lambda'$ and the function $\phi$ [6]

$$\varepsilon'(\xi) = \lambda' \nabla \phi(\sigma, \chi) \quad \text{and} \quad \lambda' \geq 0, \quad \phi(\sigma, \chi) \leq 0, \quad \lambda' \phi(\sigma, \chi) = 0
$$

(6)

where the dot above the symbol denotes the derivative with respect to the time variable, i.e., $\dot{\xi} = \partial \xi / \partial t$. Condition is also consistency condition according to which for $\dot{\phi} = 0$, plastic loading takes place if $\dot{\phi} = 0$ while unloading appears if $\dot{\phi} \leq 0$. The flow rule can also be described using maximal plastic work principle for the generalized stresses [6]

$$\varepsilon(u(x)), \sigma(x) - \sigma(x), C^{-1}\sigma(x) - \phi(x), H^{-1}\chi(x) =$$

$$\max_{(\varepsilon, \sigma) \in K} \{\varepsilon(u(x)), \sigma(x) - \sigma(x), C^{-1}\sigma(x) - \phi(x), H^{-1}\chi(x)\}
$$

(7)

$C^{-1}$ and $H^{-1}$ denote the compliance tensor and the inverse tensor to the hardening tensor, respectively.
We shall consider the following contact problem: find the generalized stress field \( (\sigma, \chi) : \Omega \rightarrow \mathbb{R}^{d \times d} \), the displacement field \( u : \Omega \rightarrow \mathbb{R}^d \), the generalized strain field \( (\varepsilon', \xi) : \Omega \rightarrow \mathbb{R}^{d \times d} \times \mathbb{R}^d \) satisfying plasticity conditions (2) - (7) as well as
\[
\text{div}(\sigma) + f_1 = 0 \quad \text{in} \quad \Omega \tag{8}
\]
\[
u = 0 \quad \text{on} \quad \Gamma_1 \quad \text{and} \quad \sigma_i = f_2 \quad \text{on} \quad \Gamma_2 \tag{9}
\]
\[
u = 0, \quad |\sigma_i| \leq \mu \sigma \quad \text{on} \quad \Gamma_3 \tag{10}
\]
\[
|\sigma_i| < \mu \sigma \Rightarrow \nu = 0 \quad \text{on} \quad \Gamma_3 \tag{11}
\]
\[
|\sigma_i| = \mu \sigma \Rightarrow \exists \lambda \geq 0, \quad \nu = -\lambda \sigma \quad \text{on} \quad \Gamma_3 \tag{12}
\]
For the unit outward normal vector \( \Gamma \) to the boundary \( \Gamma \) normal and tangential components of the displacement field \( u \) are denoted by \( \{0\} \) \( u = u + v = u \cdot v, \quad i = 1, \ldots, d \), and by \( u = u - u \cdot v \), respectively. Similarly normal and tangential components of the stress field \( \sigma \) are denoted by \( \sigma_i = \sigma \cdot v \) and by \( \sigma_i = \sigma \cdot v - v \cdot \sigma \), respectively. \( \mu \) is the friction coefficient and \( |\cdot| \) denotes the Euclidean norm. System (2)-(12) governs the elasto-plastic bilateral contact problem with Coulomb friction. We transform the system (8)-(12) into the dual variational form \([3, 6, 9]\). The yield function \( \varphi \) in (5) is chosen as von Mises function. Using the regularization of the projection operator \( P_{\kappa} \) on the set \( (5) \) as well as the regularization of the indicator function of this set as well as regularization of the friction conditions (11)-(12) dependent on a real parameter \( \beta = (\sigma, \alpha, \rho) > 0 \) the elasto-plastic contact problem (2)-(12) is equivalent to the following system of nonlinear coupled equations in the domain \( \Omega \): find \( (u, \sigma, \chi) \) satisfying
\[
\int_\Omega \varphi_i (\tau - \sigma) \, dx + \int_\Omega \chi_i (\eta - \chi) \, dx - \int_\Omega \varepsilon_i (u_i) : (\tau - \sigma) \, dx + \nabla \cdot (\varphi_i (\tau - \sigma)) = 0 \quad \text{for all} \quad (\eta, \tau) \tag{13}
\]
\[
\int_\Omega \varepsilon_i (v_i) : \sigma_i (u) + \int_\Omega \nabla j_i (v, w, u) \, ds = \int_\Gamma f_1 v \, ds + \int_\Gamma f_2 v \, ds \quad \text{for all} \quad v \tag{14}
\]
### 3. Shape Optimization Problem

Let us formulate the shape optimization problem for the state system (13)-(14). The objective functional \( J(\cdot) \) is assumed to depend on the solution \( u(\Omega) \) to the state system (13)-(14) in the domain \( \Omega \). In general this functional is sum of the domain and/or boundary integrals, i.e.,
\[
J(\Omega) = \int_\Omega \varphi_i (u) \, dx + \int_\Gamma \tilde{\varphi}_i (u) \, ds \tag{15}
\]
where the integrand functions \( \varphi_i : \mathbb{R}^d \rightarrow \mathbb{R} \) in the domain \( \Omega \) and \( \tilde{\varphi}_i : \mathbb{R}^d \rightarrow \mathbb{R} \) on boundary \( \Gamma \) depend also on the solution \( u = u(\Omega) \) to the state system (13)-(14). These integrand functions are assumed smooth enough and their first derivatives are bounded for every \( u \). Each admissible domains \( \Omega \) is assumed to be contained \([2]\) in the open and bounded hold-all domain \( \Omega \subset \mathbb{R}^d \). The boundary \( \Gamma \) of the domain \( \Omega \) is usually divided into fixed and movable parts. Therefore the set \( U_w \) of the admissible domains is written as
\[
U_w = \{ \Omega \subset \mathbb{R}^d : \Omega \subset D, \Omega \text{ is suitable regular and satisfies imposed constraints } \kappa (\Omega) \leq 0 \} \tag{16}
\]
where \( g \) is a given function. Consider the following shape optimization problem: find domain \( \Omega^* \in U_w \) minimizing the objective functional (15) on the set of admissible domains (16), i.e.,
\[
J(\Omega^*) = \min \{ J(\Omega) : \Omega \in U_w \} \tag{17}
\]
In order to find minimum of the objective functional (15) first we need to calculate its derivative with respect to the design variable \( \Omega \), i.e., its shape derivative. We shall calculate it using the Lagrangian as well as material derivative frameworks \([2]\). The adjoint variables \((\zeta, \rho, \eta)\) associated with the state variables \((u, \sigma, \chi)\) can be retrieved by differentiating the Lagrangian with respect to \( u \) and \((\sigma, \chi)\) in the directions \( v \) and \((\eta, \tau)\). It gives the system of adjoint equations satisfied by the variables \((\zeta, \rho, \eta)\)
\[
\int_\Omega \frac{d\varphi_i (u)}{du} (v_i) \, dx + \int_\Gamma \frac{d\tilde{\varphi}_i (u)}{du} (v_i) \, ds - \int_\Omega \varepsilon_i (v_i) : (p - \sigma_i) \, dx + \int_\Omega \varepsilon_i (v_i) : \sigma_i (v_i) \, dx + \int_\Gamma \nabla j_i (v, w, u) \, ds = 0 \quad \text{for all} \quad v \tag{18}
\]
all for \( v \) as well as
\[
\int_\Omega \zeta_i : C^{-1} (p - v_i) \, dx + \int_\Omega \eta_i : H^{-1} (q - \eta_i) \, dx - \int_\Omega \varepsilon_i (u_i) : (p - \eta_i) \, dx + \int_\otimes \left[ (\tau_j - P_{\kappa} (\eta_j)) : p + (\eta_j - P_{\kappa} (\eta_j)) : q \right] \, ds = 0 \quad \text{for all} \quad (\eta, \tau) \tag{19}
\]
Therefore using (18)-(19) we obtain the shape derivative of the cost functional (15) as equal to the
following boundary integral

\[
J'(\Omega)(\xi) = \int_\Gamma (\xi \nabla u) \cdot \nu ds - \int_\Gamma (H_w \nabla^2 u + \nabla (H_w \nabla u)) \cdot \nu ds + \int_\Gamma (H_w \nabla \phi \nabla (H_w \nabla u)) \cdot \nu ds - \int_\Gamma (f + f(z)) \phi ds
\]

where \((u, \sigma, \chi)\) and \((z, p, q)\) are solutions to the state system (13)-(14) and the adjoint system (18)-(19), respectively.

4. Numerical Methods

We will use the level set function \([13]\) to define and to update the shapes of the admissible domains \(\Omega\). All admissible shapes \(\Omega\) are contained in a hold-all open bounded domain \(D \subset R^2\). Let us introduce the notion of the level set function \(\Psi(\cdot), R^{d+1} \rightarrow R\):

\[
\Psi(x) = 0 \text{ if } x \in \Gamma \cap D, \quad \Psi(x) < 0 \text{ if } x \in \Omega, \quad \Psi(x) > 0 \text{ if } x \notin \Omega.
\]

The boundary \(\Gamma\) of the domain \(\Omega\) is located as a set of points in \(D\) where the level set function \(\Psi(x) = 0\). The normal vector \(\nu\) and the mean curvature \(H_w\) of the boundary \(\Gamma\) in terms of the level set function are equal to \(\nu = \nabla \Psi / \| \nabla \Psi \|\) and \(H_w = \text{div}(\nu)\). The level set algorithm starts from an initial shape domain \(\Omega_0\). During the optimization process a sequence of domains \(\Omega_i\), \(i = 1, 2, \cdots, N_i\), is generated. Each domain \(\Omega_i\) is characterized by its level set function. Therefore the shape evolution process may be associated with fictitious time variable \(t\), \(0 \leq t \leq T\). \(T\) is a given real constant, and the level set function is dependent on space and time variables, i.e., \(\Psi = \Psi(t, x)\). In classical setting the evolution of the level set function from domain \(\Omega_0\) to domain \(\Omega_{i+1}\) is governed the Hamilton-Jacobi equation in time interval \(\lbrack 0, T \rbrack\) and in the domain \(D\)

\[
\frac{\partial \Psi}{\partial t} + V \cdot \nabla \Psi = 0, \quad \Psi(0, x) = \Psi_0(x)
\]

where function \(\Psi_0(x)\) determining the shape of the initial domain \(\Omega_0\) is given. The normal velocity of the domain boundary \(V(x)\) is identified with the shape derivative (20) of the objective functional (15) in the direction \(\xi\). Since the shape derivative is given as boundary integral the normal velocity \(V\) has to be extended on the whole computational domain \(D\). This extension is usually associated with regularization of velocity \(V\). The equation (21) is solved usually by an explicit second order upwind scheme on a Cartesian grid of design domain \(D\) with Neumann boundary conditions. In order to ensure the stability of this scheme the time step has to satisfy CFL condition relating the length of time and space discretization steps. Since during the iteration process the level set may become too flat or too steep in order to regularize it periodic re-initialization of the level set function is performed. For other types of the level set methods such as binary or piecewise constant see \([13]\).

Both the state and the adjoint boundary value problems (13)-(14) as well as (18)-(19) are discretized using bilinear quadrangular finite elements. Ersatz material approach \([2, 9, 10]\) is used to avoid meshing problems with the shapes of domains \(\Omega\) and singularity of the stiffness matrix. This approach consists in filling the domain \(D \setminus \Omega\) with a weak material characterized by law value of Young modulus and mimicking void. The discretized state and adjoint equations are solved numerically using semi-smooth Newton method \([5]\).

5. Numerical Example

Topology optimization problem for a body occupying two-dimensional domain \(\Omega \subset R^2\) in bilateral contact with the foundation has been solved numerically. The functional (15) with \(\Psi(u) = 0\) and \(\Psi(u) = \sigma(u) \cdot v\) is chosen as the objective functional. The aim of the optimization problem is to minimize the contact stress on the boundary \(\Gamma_s\). The constraint function \(g(u)\) in (16) is equal to volume constraint, i.e., \(g(u) = \text{Vol} (\Omega) = \int_\Omega dx - V_s\). The domain \(\Omega \subset R^2\) is chosen as follows (see Figure 1 for details)

\[
\Omega = \{ (x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8, 0 \leq x_2 \leq 4 \}\] (22)
Remark, the domain $\Omega$ and its boundary $\Gamma$ depend on the function $z(x_3)$. The domain $\Omega$ is filled with material characterized by the Young moduli: $E_1 = 10 \cdot E_2$, $E_2 = 1 \cdot E_3$, and $E_3 = 2.1 \cdot 10^5$ MPa (see Figure 2). The Poisson’s ratio is $\nu = 0.3$. The shear and dilation moduli are equal to $8 \cdot 10^5$ MPa and $1.1 \cdot 10^5$ MPa. The yield stress $\sigma_y = 3674$ MPa. The hardening parameters are equal to $k_1 = 1 \cdot 10^5$ MPa and $k_2 = 0$ MPa. The body is loaded by the boundary traction $f_2 = -6.5 \cdot 10^7$ N along the boundary $\Gamma_2$, the body force $f_1 = 0$. The domain $D$ is a rectangle $[0.8] \times [0, 4]$. This domain is divided into $80 \times 40$ grid. Figure 3 displays the obtained optimal topology. In a case of elasto-plastic materials the mass of the structure is larger than in a case of elastic model. The algorithm tries to avoid the generation of plastic zones which are less rigid and induce larger displacements. Von Mises effective stress concentration areas appear close to the contact zone.

6. Conclusions

This study shows that presented approach based on the application of the level set technique can be applied to solve numerically a topology optimization problem for bodies in contact using nonlinear small strain elasto-plastic material model with linear kinematic hardening rather than the linear material model. It allows to formulate optimality conditions for this type of nonlinear problems and it is capable of generating topologies that generates minimum contact stress.

References

EQUIVALENT STATIC DISPLACEMENTS METHOD FOR NONLINEAR OPTIMIZATION OF CONTACT FORCE

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Abstract

The contact force optimization problem is extensively exists in engineering. For example, the contact force between seat and human body is an important index for evaluating seat comfort. The contact force-deflection curves of moving deformable barrier in crash test should satisfy the requirements of regulation for automotive crashworthiness. However, there are few effective methods to solve contact force optimization due to the high nonlinearity. This paper introduces the equivalent static displacements (ESD) method to solve the contact force optimization, which is a dual method of the existed equivalent static loads (ESL) method. The equivalent static displacements are defined as enforced displacements whose reaction forces in linear static analysis are equal to the contact forces in nonlinear analysis. Firstly, nonlinear analysis with contact condition is conducted to calculate the real contact forces. Then, apply the contact forces to structure as external linear static loads and calculate the nodal displacements at contact surface, which are the equivalent static displacements. Finally, the ESD is applied to structure as linear static enforced displacements. In this condition, the reaction forces of ESD are equal to the real contact forces. And static structural optimization can be performed to optimize the reaction forces. Therefore, the ESD method can be used in structural design with contact problem. An engineering example is carried out to verify the efficiency of the ESD method.

Key words: Structural Optimization, Contact Force Optimization, Equivalent Static Displacements

1. Introduction

The contact loading condition exists extensively in the structural design field. The magnitude and distribution of contact forces commonly have an important influence on structural performance. For example, the contact pressure distribution between human body and seat is used to measure the seat comfort. Crash force-displacement curve response of automobile body is an important index for evaluating crashworthiness design, as shown in Figure 1. On the premise of energy absorption, the crash force should be as small as possible to prevent injury to passengers. Moreover, in medical implants design, stress concentration between implants and human tissues will cause pain and damage in patients.

Figure 1. Automobile front impact and force-displacement curve

Therefore, the structural optimization considering contact force response is of practical significance. Fancello et al. perform shape optimization by a direct modification of the geometry through B-spline curves [1]. Herskovits et al. apply a bi-level method to solve shape optimization problems [2]. Li et al. solved the elastic contact shape optimization under multiple load cases [3]. Contact pressures or stresses are considered in the above shape...
optimization methods. Klarbring et al. solved the topology optimization of static contact between truss and rigid body [4]. Fanello solved topology optimization under static contact stress constraints with minimum mass as objective [5]. Lawry et al. combined the level set method and XFEM method to solve the shape and topology optimization of two elastic bodies [6]. Luo et al. modeled frictionless contact with hypothetical nonlinear spring and solved the topology optimization of static contact between the hyperelastic structure and rigid body [7]. Lawry et al. carried out topology optimization considering structural force-displacement curve response under finite strain, frictionless and static contact conditions [8]. Zhang and Niu assumed that contact force is a linear function of contact clearance to solve the shape optimization, considering nodal contact forces, elemental contact pressures, mass and displacements responses[9]. In addition, Niu and Zhang developed a topology optimization method for the uniformity of contact pressures by introducing the variance constraint of contact pressures [10].

The above mentioned researches are all concentrated in static contact load cases. And a few practical engineering examples have been realized. Furthermore, the structural optimization considering dynamic contact force responses, such as crash force constraints, has not been solved. For nonlinear dynamic optimization, Kang et al innovatively proposed the ESL method [11]. The nonlinear dynamic load cases are converted into multiple linear static load cases in ESL method. The structural displacements calculated from static load cases are equal to those calculated from nonlinear dynamic analysis. In addition, the structural optimization under static load cases can be easily solved by mature algorithms or commercial software. By using ESL method, linear dynamic [12], multi-body dynamic [13-14], nonlinear static [15-17] and nonlinear dynamic [18-19] optimization problems have been solved. Some engineering examples [20-21] have also verified this method.

Force and displacement are dual physical quantities. Inspired by this idea, the dual method of the ESL method, equivalent static displacements (ESD) method is proposed in this paper. The contact forces in static or dynamic loading conditions are converted into linear static enforced displacements. The anti-forces of these enforced displacements are equal to the real contact forces. Therefore, the contact forces optimization can be solved by conducting linear static optimization with reaction forces response.

This paper is organized as follows. In section 2, the details of the ESD method are presented. In section 3, an engineering example considering crash force response is carried out to verify the proposed ESD method. Finally, conclusions are summarized in section 4.

2. Equivalent Static Displacements Method

2.1 Definition of ESD

For static contact problem, the ESD is defined as a set of static enforced displacements, whose reaction forces are equal to the real contact forces. For dynamic contact problem, an ESD is made for each time step of dynamic contact analysis, as illustrated in Figure 2. The reaction forces calculated from the ith ESD (the s, load case) are equal to the real contact forces calculated from the dynamic contact analysis at the ith time step. Therefore, the dynamic contact loading condition is converted into multiple static loading conditions with ESDs. And the total number of time steps are equal to the total number of ESDs.

![Figure 2. Definition of ESD](image-url)
For static contact problem, or each time step of dynamic contact problem, the calculation process of ESD is presented in Figure 3. This process is same for structure A and B. Therefore, structure A is taken as an example to illustrate the detail calculation process. The steps are as follows:

Step 1: Conduct contact nonlinear analysis to calculate the real contact forces \( f_{\text{cont}} \), as shown in Figure 3(a).

Step 2: The original external load \( f_e \) and fixed boundary keep unchanged. Apply the contact forces \( f_{\text{cont}} \) to the structure as static loads, as shown in Figure 3(b). Then calculate the structural displacements vector \( \mathbf{u}_e \) under this linear static loading condition.

Step 3: \( \mathbf{u}_e \) consists of displacements of all nodes, \( \mathbf{u}_e^i \) in Figure 3(c) denotes the ESD, which consists of the displacements of nodes on the original contact surface. Take these values from \( \mathbf{u}_e \) and apply them to the original location as enforced displacements, as shown in Figure 3(c). Meanwhile, the nodes on other location are still not applied forces or constraints. In this loading condition, the reaction forces of \( \mathbf{u}_e^i \) are equal to \( f_{\text{cont}} \).

![Figure 3. Calculation process of ESD](image)

### 2.2 Proof of the Equivalence Between Contact Forces And Reaction Forces

It is not obvious that the reaction forces of ESD are equal to the real contact forces. Therefore, taking the structure A (shown in Figure 3(b) and Figure 3(c)) as study object, this section proof the equivalence between contact forces and reaction forces. In order to simplify the proof process, the external forces \( f_e \) are omitted, which has no effect on the proof process. The simplified transformation process from contact forces to ESD is illustrated in Figure 4.

As shown in Figure 4(a), the contact forces are applied to the structure as linear static loads. The finite element equation is as follows:

\[
K \begin{bmatrix} \mathbf{u}_c & \mathbf{u}_f \end{bmatrix}^T = \begin{bmatrix} \mathbf{f}_{\text{cont}} & \mathbf{f}_{\text{fix}} & \mathbf{0} \end{bmatrix}^T
\]

(1)

where \( K \) is the stiffness matrix, \( \mathbf{u}_c \), \( \mathbf{u}_f \), and \( \mathbf{u} \) are the displacement vectors of contact region, fixed boundary and other region, respectively; \( \mathbf{f}_{\text{cont}} \) is the contact force vector, which is calculated from contact nonlinear analysis; \( \mathbf{f}_{\text{fix}} \) is the reaction force vector of fixed boundary. \( \mathbf{0} \) means that in other region, the external load is zero. In Eq. (1), \( \mathbf{u}_c \), \( \mathbf{u} \) and \( \mathbf{f}_{\text{fix}} \) are unknown, \( \mathbf{f}_{\text{cont}} \) and \( \mathbf{u}_f \) are known.

![Figure 4. The simplified transformation process from contact forces to ESD](image)

Figure 4(b) illustrates that the ESD (\( \mathbf{u}_{\text{eq}} \)) is applied to the nodes on the original contact surface. The finite element equation of this loading condition is:

\[
K \begin{bmatrix} \mathbf{u}_{\text{eq}} & \mathbf{u}_{\text{fix}}' \end{bmatrix}^T = \begin{bmatrix} \mathbf{f}_{\text{ESD}} & \mathbf{f}_{\text{fix}}' & \mathbf{0} \end{bmatrix}^T
\]

(2)

where \( \mathbf{u}_{\text{eq}} \) is the ESD, which is equal to \( \mathbf{u}_c \) calculated in Eq. (1); \( \mathbf{u}_{\text{fix}}' \) is the displacement vector of the fixed boundary.
Since the fixed boundary is unchanged, $\mathbf{u}'_{\text{fix}} = \mathbf{u}_{\text{fix}}$; $\mathbf{u}'$ is the displacement vector of the other region of structure; $\mathbf{f}_{\text{ESD}}$ and $\mathbf{f}'_{\text{ESD}}$ are the reaction force vectors of $\mathbf{u}_{\text{eq}}$ and $\mathbf{u}'_{\text{fix}}$, respectively, 0 means that in other region, the external load is zero. In Eq. (2), $\mathbf{u}'$, $\mathbf{f}_{\text{ESD}}$ and $\mathbf{f}'_{\text{ESD}}$ are unknown, $\mathbf{u}_{\text{eq}}$ and $\mathbf{u}'_{\text{fix}}$ are known.

To prove the reaction forces of ESD are equal to the real contact forces, we should prove $\mathbf{f}_{\text{ESD}} = \mathbf{f}_{\text{cont}}$. However, whether $\mathbf{u}'$ is equal to $\mathbf{u}$ is unknown. Substitute $\mathbf{u}' = \mathbf{u}$ into Eq. (2), $\mathbf{f}_{\text{ESD}} = \mathbf{f}_{\text{cont}}$ is obviously satisfied. In addition, linear static problem has one and only one solution. Therefore, $\mathbf{u}' = \mathbf{u}$ and $\mathbf{f}_{\text{ESD}} = \mathbf{f}_{\text{cont}}$ is the unique solution of Eq. (2).

2. 3 Procedure of the ESD Method

The overall process of the proposed ESD method consists of four parts as shown in Figure 5. The contact forces are obtained in part 1 and used to calculate the ESD in part 2. Then, in part 3, linear static optimization is carried out using ESD. In part 4, the optimization result is utilized to generate an updated finite element (FE) model. Since the design is changed, the characteristics of the structure are changed and the above process should be conducted again. Therefore, the entire optimization procedure circulates until the convergence condition is satisfied.

![Figure 5. Schematic of the ESD method](image)

The detail procedure of the ESD method is illustrated in Figure 6, where $k$ is the cycle number, $\mathbf{x}$ is the design variable vector, $\mathbf{u}$ is the displacement vector, $\mathbf{M}$ and $\mathbf{K}$ are the mass matrix and stiffness matrix, respectively, $\mathbf{f}(t)$ is the external load vector, $\mathbf{f}_{\text{cont}}$ is the contact force vector, $\mathbf{u}_{\text{eq}}$ is the ESD, $g_i$ is the $i$th constraint condition. The detail steps of this method are as follows.

Step 1: Set the initial values (cycle number $k=0$, design variable vector $\mathbf{x}^{(0)} = (x_1, x_2, \ldots, x_n)^{(0)}$, a small value $\varepsilon$ as permitted convergence error).

Step 2: Perform nonlinear contact analysis to calculate the contact force vector $\mathbf{f}_{\text{cont}}$.

Step 3: Calculate the ESD according to the process illustrated in section 2.1.

Step 4: Apply ESD to the original contact surface as enforced displacements. Then perform the linear static optimization and get the optimal design variable vector $\mathbf{x}^{k+1}$.

Step 5: If convergence condition, such as Eq. (3), is satisfied, terminate the procedure. Otherwise, update the design variables $\mathbf{x} = \mathbf{x}^{k+1}$ and turn to Step 2.

$$\| \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)} \| \leq \varepsilon$$  \hspace{1cm} (3)

![Figure 6. Optimization procedures of the ESD method](image)
The above procedure is independent of the type of design variables. Therefore, all of the size, shape and topology optimization considering contact forces can be realized by using ESD method. In addition, since the kernels of contact analysis and linear static optimization are not involved, the contact types, such as frictionless or frictional contact, are unlimited, and commercial software can be utilized. High cohesion and low coupling characteristics are the advantages of the ESD method.

3. Size Optimization Examples

An aluminum side frame structure from automobile body is employed to test the peak contact force optimization. As illustrated in Figure 7, the side frame can protect the passengers from injury in automobile side impact. The finite element model of crash loading condition is shown in Figure 8. Commercial software LS-Dyna is used for crash analysis, and OptiStruct is used for calculating ESD and conducting linear static optimization.

![Figure 7. Schema of side frame](image)

The side frame structure consists of aluminum tubes, with material density 2730 kg/m³, elastic modulus 64 GPa, Poisson’s ratio 0.3 and yield stress 200 MPa. Aluminum tubes can be directly extruded to form complex cross-sectional, as presented in Figure 9. And the thickness of each edge, such as ti in Figure 9, can be independent designed variable. Figure 10 illustrates the detail definitions of thickness design variables in this problem. The size optimization formulation is to search the minimal mass, constrained with peak crash force displacements:

\[
\text{Find } \ t = (t_1, t_2, \ldots, t_{20}) \\
\text{min } \text{Mass} \\
s. \ t. \\
1 \text{ mm} < t_i < 10 \text{ mm} \\
270 \text{ kN} < F_{\text{max}} < 290 \text{ kN}
\]

where ti is the ith design variable, as shown in Figure 11. Fmax is the peak crash force. The permitted convergence error shown in Eq. (3) is set to 0.02.

The convergence criterion is satisfied within 7 cycles. The curves of mass and peak crash force vs. iteration numbers are shown in Figure 11(a) and Figure 11(b), respectively. The optimization results of design variables and responses are presented in Table 1 and Table 2, respectively. The mass is reduced from 21.67 kg to 17.71 kg. The peak crash force is reduced from 337.2 kN to 270.2 kN. The constraint conditions are satisfied.

![Figure 8. Rigid pillar crash loading condition](image)

![Figure 9. Aluminum tubes and their cross sectional shape](image)

![Figure 10. Definition of design variables](image)
4. Conclusion

The equivalent static displacements method is proposed to solve the structural optimization considering contact forces response. The nonlinear contact forces are transformed into linear static enforced displacements in ESD method. The reaction forces of ESD are exactly equal to the real contact forces. Then, linear static optimization considering reaction forces response is conducted to replace the nonlinear contact force response optimization. Therefore, structural optimization using ESD does not directly use nonlinear sensitivity information. Both static and dynamic contact forces optimization can be solved by using ESD method. A crash force optimization example is carried out to verify the effectiveness of the ESD method. Since the ESD is independent of the type of design variables, it can be used in size, shape and topology optimization. Meanwhile, ESD method is unlimited for contact types, such as frictionless or frictional contact, node to node or surface to surface contact, because the kernel of the contact analysis procedures is not involved. In addition, ESD method has the characteristics of “high cohesion and low coupling”. Only contact forces are transformed to enforced displacements. Contact analysis and static optimization can be realized by professional commercial software.
Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 51575226).

References

OPTIMAL APPLICATION OF TRIGGERS FOR THE DEFORMATION OF CRASH STRUCTURES UNDER AXIAL COMPRESSION WITH CONSIDERATION OF GEOMETRICAL IMPERFECTIONS

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Abstract

Extruded profile structures are widely used as efficient energy absorbers in vehicle front and rear ends throughout the automotive industry. Especially for crash structures under axial compression the usage of physical trigger mechanisms is state of the art to robustly initiate and maintain the desired deformation modes. To simulate these axially loaded crash structures, imperfections are needed to avoid unrealistic deformation modes caused by an ideal finite element model. Furthermore stochastic imperfections can be used for robustness evaluations. Hence an automatic model creation process has been developed which allows an application of geometrical imperfections on any given extruded profile structure. This is done by extending GRAMB (GRAph based Mechanics Builder) which is part of the Graph and Heuristic based Topology Optimization (GHT). Furthermore a parametrized and graph related mesh modification and morphing process has been implemented, which allows the application of various physical triggers such as indents, cutouts and chamfers on profiles generated by GRAMB. The application of the physical triggers has been investigated in parameter optimizations for a longitudinal member in a front crash load case. The crashworthiness of the front end has been improved and the robustness of the optimal designs was investigated.

Key words: Topology Optimization, Crash Load Cases, Profile Structures, Geometrical Imperfections, Triggers for Crash Structures

1. Introduction

Throughout the automotive industry, extruded profile structures are broadly used as efficient energy absorbers in vehicle front and rear ends. Although a lot of theoretical, experimental and numerical investigations have been performed over the years, there is still research work ongoing, particularly when it comes to the (topology) optimization of these structures in highly non-linear crash load cases (e.g. [1-2]).

The present work is intended as a basis for topology optimization of axially loaded structures with the Graph and Heuristic based Topology Optimization (GHT) [3]. Therefore the state of the art in the automotive industry such as the usage of physical triggers for the robust initiation and maintenance of desired deformation modes and crash characteristics has to be implemented in the automatic model creation process. Furthermore the simulation of these axially loaded crash structures requires the application of imperfections in order to avoid unrealistic deformation modes caused by an ideal FE model and to investigate the robustness regarding the structural response. Hence the GRAPb based Mechanics Builder (GRAMB) which is part of the GHT is extended. GRAMB is used to extrude profile cross sections defined by a mathematical graph along a spline in order to automatically generate a finite element (FE) mesh. For the numerical studies in this work the explicit FE solver PAM-CRASH (v2017) is used.

In this work geometrical imperfections are defined as model modifications which are only used in the FE simulation in order to acquire a better accordance with real world observations. In comparison, physical triggers are defined as imperfections that are utilized both on real world and simulation components to influence the crash characteristic of the component and to robustly establish and maintain a desired mode of deformation.
2. Geometrical Imperfections

2.1 Background

For the progressive buckling of square profiles four types of modes are described by [4]: extensional, symmetric and two asymmetric modes. The extensional mode (axis symmetric folding pattern—compare Figure 1) absorbs more energy than the other modes, but it can barely be observed in real world tests. Instead the other three modes predominate. The simulation of the buckling behavior of straight profiles is challenging since small variations in boundary conditions, material properties and geometry can lead to a considerable variation in the structural behavior and therefore the energy absorption [5]. The ideal geometry description as well as ideal material data and ideal boundary conditions, which are usually used in the FE simulation, often lead to extensional modes for squared tubes [6].

To get more realistic modes in the FE simulation, several measures can be taken. Most of these replicate imperfections found in real world specimens and test procedures; Stochastic material data [7], stochastic loading conditions, stochastic wall thickness distributions [7] or stochastic profile wall deflections (nodal displacements) [5, 8]. Another measure often taken is the use of deterministic triggers (e.g. wall deflection at the top section of the profile) to enforce a profile collapse in the desired mode [9].

Since the method described in this paper should be used with any profile cross section topology created by GRAMB, where the desired mode is often unknown, a deterministic trigger for a specific mode is not an option. Furthermore since only the profile models themselves can be manipulated by GRAMB, stochastic loading or boundary conditions are unsuitable as well. In [6] found that the deflection of the profile walls has the biggest influence on the crushing mode compared to wall thickness and wall length deviation. Therefore a stochastic modification based on wall deflection was found to suit the requirements best to enable realistic modes and robustness investigations for unknown profile cross sections in an automatic model creation process.

2.2 Simulation and Application of Stochastic Fields

The approach for generating the stochastic fields used by [5] and [7] is based on research by [10] where methods for the simulation of multi-dimensional Gaussian stochastic fields by spectral representation are described. While [5] and [7] use a two-dimensional, univariate Gaussian field (2D-1V) for their respective work, a new approach is described in this work which is applicable to generate smooth fields for more complex profile structures (e.g. with inner profile knots and walls) as well as an implementation in the automatic model creation process with GRAMB. The first term in “3D-YV” represents the input of the stochastic field (e.g. 2D or 3D FE node coordinates), while the second term describes the number of stochastic field output variables (1V-univariate).

For this purpose, two 3D-1V fields as described in [10] are used. The output of each field is used for the nodal displacement in both directions (x1 and x2) of the cross section plane respectively (Figure 1). They are computed by a series of cosines Eq. 1, which is computationally expensive but provides flexibility with non-equidistant meshes. For Eq. 1, random numbers (\( \Phi_{4, \alpha} \)) between 0 and 1 have to be generated. Further information regarding Eq. 1 can be found in [10]. The simulation of the stochastic fields mathematically requires an infinite series of cosines. Since this cannot be established in an automated process which pursues short calculation times, a tradeoff has to be accepted and the number of sums is limited to a finite value (\( N_1 = N_2 = N_3 = 25 \)).

\[
f^{(c)} (x_1, x_2, x_3) = \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} \left[ A_{n_1, n_2}^{(1)} \cos(\kappa_{n_1} x_1 + \kappa_{n_2} x_2 + \Phi_{4, n_1}^{(1)}) + A_{n_1, n_2}^{(2)} \cos(\kappa_{n_1} x_1 + \kappa_{n_2} x_2 + \Phi_{4, n_1}^{(2)}) \right] + \left[ A_{n_1, n_2}^{(1)} \cos(\kappa_{n_1} x_1 + \kappa_{n_2} x_2 + \Phi_{4, n_1}^{(1)}) + A_{n_1, n_2}^{(2)} \cos(\kappa_{n_1} x_1 + \kappa_{n_2} x_2 + \Phi_{4, n_1}^{(2)}) \right]
\]

A power spectral density (PSD) function for 3D-1V fields is described in [11]. This function is part of the four \( A_{4, \alpha} \) terms in Eq. 1. As a tradeoff for the number of sums has to be accepted, the PSD function is modified in a way, that the output standard deviation of the stochastic fields corresponds with the input standard deviation.

\[
S_{ij} (\kappa_1, \kappa_2, \kappa_3) = \sigma_r \frac{b_1 b_2 b_3}{16 \pi} \exp \left[ - \left( \frac{b_i \kappa_j}{2} \right)^2 - \left( \frac{b_j \kappa_j}{2} \right)^2 - \left( \frac{b_i \kappa_j}{2} \right)^2 \right]
\]
The PSD function used is stated in Eq. 1, where \( \sigma \) is the standard deviation of the stochastic field. The parameters \( b_1 \), \( b_2 \) and \( b_3 \) are proportional to the correlation length of the stochastic fields in respective \( x_1 \), \( x_2 \) and \( x_3 \) directions (Figure 1). Since \( x_1 \) and \( x_2 \) are directed in the cross section plane, the same parameter \( b_{\text{local}} \) is used for \( b_1 \) and \( b_2 \). The parameter \( b_3 \) in extrusion direction is renamed to \( b_{\text{ext}} \). The output values of both fields are calculated with Eq. 1, where the first field provides the stochastic displacement in \( x_1 \)-direction, and the second field the ones in \( x_2 \)-direction respectively. The input values for the simulation of both stochastic fields are the three dimensional FE node coordinates in the \( x_1-x_2-x_3 \) coordinate system. In this work the parameters described by [12] are used (Table 1).

<table>
<thead>
<tr>
<th>Imperfection</th>
<th>( \sigma ) in mm</th>
<th>( b_{\text{local}} ) in mm</th>
<th>( b_{\text{ext}} ) in mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local</td>
<td>0.07</td>
<td>40</td>
<td>400</td>
</tr>
</tbody>
</table>

The method developed in this work can also be applied to profiles extruded along a three-dimensional spline. The \( x_1-x_2-x_3 \) coordinate system is therefore translated and rotated for each spline section so that the field is smoothly aligned throughout the whole extrusion length.

Figure 1 shows example results for a square aluminum extrusion profile (AA6XXX T6, 80 x 80 x 400 mm, 1 kg) with and without applied stochastic fields in a simple drop hammer test. It can be seen that the dominant mode of deformation changed after the application of the stochastic fields from extensional mode to symmetric mode. For a series of ten tests with individual stochastic fields an increase of 13.3% for the intrusion can be observed compared to the ideal FE geometry.

3. Physical Triggers

3.1 Background

Physical triggers are state of the art for automotive applications in order to initiate and maintain a defined and robust crash behavior. The topic has been studied intensively by many researchers. The work by [13] provides a broad overview over publications related to geometrical and material triggers for axially loaded crash structures. The measures described in literature contain prebuckling, corner indentations, triggering dents, corrugation of tubes, cutouts, chamfering, tube wrapping and many more.

Not all of the physical triggers listed above meet the requirements for an automotive application which seeks for cost efficient manufacturing while still being able to influence the crash characteristic significantly. As a result, indentations as well as various cutouts are the most common physical triggers for aluminum profiles in automotive front end applications. Hence these triggers are implemented in the automatic model creation process. Another trigger that seems industrially feasible is the chamfering of extruded profiles.
3. 2 Modelling of Physical Triggers

The application of the physical triggers is implemented in the automatic model creation process with GRAMB as a downstream mesh modification and morphing process after the completion of the initial meshing. Own techniques were developed since a connection between the existing profile mesh geometry and the mathematical graph used by GRAMB should be established in order to integrate the physical triggers in the optimization process of the GHT. All triggers are defined by their relative position on the extrusion spline (from 0 to 1) and within the profile cross section as well as other trigger specific parameters such as the trigger dimensions. The triggers implemented are profile corner and wall indents, profile corner and wall cutouts as well as end- and mid-section chamfers. Example applications of the physical triggers are illustrated in Figure 2 (a). It should be noted that for the chamfering triggers not only the FE nodes are moved, but also the related element thicknesses are reduced. Figure 2 (b) shows the specific modelling parameters for the wall indent trigger.

![Figure 2. (a) Example application of physical triggers (b) Trigger specific parameters for wall indent trigger](image)

4. Example Optimization of Trigger Parameters in a Crash Load Case

4.1 FE Model

The crash test investigated in this work is the RCAR Front Impact [14]. In this load case a vehicle impacts at a speed of 15 km/h into an oblique (10°) rigid barrier with a 40% overlap. Instead of a full vehicle, a simple generic model of a vehicle front end is used (Figure 3). It consists of an aluminum cross member facing the barrier with two aluminum longitudinal members tied to it. The material is AA6XXX T6 with a yield strength of 280 MPa.

![Figure 3. Example application of physical triggers](image)

The longitudinal members are connected via a rigid body definition with a point mass which inherits the weight (1500 kg) and the initial velocity (15 km/h) of the vehicle. The longitudinal members have a length of 500 mm and a rectangular profile geometry (130 mm x 90 mm) with a wall thickness of 2 mm which is not changed during the optimizations. The triggers are only applied to the longitudinal member on the side of the barrier overlap.

4.2 Optimization Setup

The parameter optimization is performed with GRAMB and the commercial software LS-Opt (v5, 21). A genetic algorithm with direct optimization [15] is used for the present optimization problem since it provides good design space exploration which is needed for the highly nonlinear structural responses of the crash simulations. The design variables used for this optimization are the positions of the physical triggers in extrusion direction as described in section 3.2.

The objective of the optimization is the minimization of the maximum contact force measured between the front end and the barrier. Peaks in the force level during the crash event are undesirable since they can result in higher loads on the passengers as well as unwanted plastic deformation of downstream vehicle structures. As a constraint, the
maximum intrusion is limited to 280 mm. To prevent the front end from sliding sideward along the barrier instead of absorbing energy by plastic deformation, a displacement constraint in y-direction is set.

4.3 Example Parameter Optimizations

In the first parameter optimization the positions of twelve wall indents are used as design variables in order to trigger homogeneous symmetric buckling in the profile. In order to reduce the number of independent design variables, the position of the first trigger is used as one independent variable. As a second independent design variable a factor for the distances between the triggers is defined. Each trigger has a counterpart on the opposite profile side and the indent trigger positions on the long and short sides are shifted against each other in order to trigger the symmetric mode.

In the second parameter optimization four cutout triggers are applied to the structure. One cutout trigger is placed at each wall of the longitudinal member. The four axial positions of the triggers are the independent variables in this optimization. This optimization setup enables asymmetric trigger patterns, although the cutout triggers have corresponding positions close to the cross member in the initial design.

The optimal designs found in each of the parameter optimizations and their structural responses are shown in Table 2 and Figure 4. Both optimizations result in a more uniform and therefore improved force-displacement curve. In comparison, the force-displacement curve of the reference structure as well as the initial designs with triggers (not shown in Figure 4) shows high force amplitudes with significant decreases of the force level in between. The reference structure buckles with an asymmetric folding pattern in the beginning which then shifts to a more symmetric one in the consecutive folds.

<table>
<thead>
<tr>
<th>Design</th>
<th>$F_{\text{max}}$ in kN</th>
<th>$I_{\text{max}}$ in mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial</td>
<td>optimum</td>
<td></td>
</tr>
<tr>
<td>initial</td>
<td>optimum</td>
<td></td>
</tr>
<tr>
<td>Reference w/o trigger</td>
<td>71.9</td>
<td>275.5</td>
</tr>
<tr>
<td>Indent wall - 2 indep., 11 dep. design variables</td>
<td>83.4</td>
<td>57.9</td>
</tr>
<tr>
<td>Cutout wall - 4 indep. design variables</td>
<td>78.5</td>
<td>57.6</td>
</tr>
</tbody>
</table>

Figure 4. Reference and optimal designs from parameter optimization (displacement scaled with factor 0.5)

As intended by the optimization setup, the optimal structures with symmetric wall indents buckle in an almost perfect symmetric mode starting from the first fold on. The trigger setup lowers the force peaks significantly by directly initiating buckling in a symmetric mode and enforcing a shorter and therefore more efficient folding wavelength than in the reference structure. The symmetric buckling mode is maintained during the whole deformation. In comparison with the initial design, the distance between the trigger positions was specially increased (indent triggers in the initial design were more packed) in order to buckle in the symmetric folding pattern throughout the whole crash event.

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For the optimization run with the cutout triggers at each side of the profile an asymmetric trigger pattern is the optimal design in terms of the achieved objective function. In contrast to the optimal symmetric wall indent design as well as to the reference design, the folding pattern observed here is more irregular. Still it results in a significant decrease of the maximum force comparable to the symmetric design. The first fold is initiated by two cutouts which reduce the first force peak effectively and induce an asymmetric pattern. A third cutout initiates the next fold which preserves the asymmetric behavior. The fourth cutout at the end of the profile is not utilized as a buckling initiator.

4.4 Robustness Analysis of Optimal Designs

Although both trigger designs initiate completely different folding patterns, the resulting force reduction is still similar. To investigate the robustness of these designs, ten calculations per design with applied stochastic fields (section 2.2) are performed. The standard deviations for the maximum force and intrusion are used as evaluation criteria for the robustness. Table 3 shows the results of this investigation. As expected, the design with the symmetric trigger pattern on its walls shows a more robust behavior regarding geometrical imperfections than the reference design and the design with the asymmetric pattern. In case of the symmetric design the defined folding pattern induced by the numerous wall indents is the cause. For the cutout trigger design only the first two lobes are directly triggered by cutouts. Therefore no defined folding pattern is enforced for the consecutive folds. It even shows a less robust behavior than the reference design without any triggers. Due to the applied imperfections, a bifurcation point can be identified in the folding pattern of the cutout design and therefore also in the structural responses. It is caused by the asymmetric trigger pattern, the inferior number of triggers compared to the symmetric design as well as potential contact points at the cutout edges.

<table>
<thead>
<tr>
<th>Design</th>
<th>$F_{\text{max}}$ in kN</th>
<th>$\text{Int}_{\text{max}}$ in mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference w/o trigger</td>
<td>71.6 0.67</td>
<td>275.5 0.70</td>
</tr>
<tr>
<td>Indent wall - 2 indep., 11 dep. design variables</td>
<td>57.9 0.13</td>
<td>272.7 0.34</td>
</tr>
<tr>
<td>Cutout wall - 4 indep. design variables</td>
<td>61.9 3.75</td>
<td>269.9 3.56</td>
</tr>
</tbody>
</table>

5. Conclusions

In this work the necessary foundations for a topology optimization of axially loaded structures with the GHT have been presented. The investigations show that the developed geometrical imperfections provide reasonable results for triggering realistic deformation modes and have potential for the evaluation of the robustness of crush structures at the end of an optimization. Furthermore the physical triggers presented in this paper have been used in parameter optimizations and lead to significant improvements in the crushworthiness of a generic front end.

Further research will focus on the development of new heuristics for the GHT in order to perform topology optimizations of axially loaded crash structures. Besides the heuristics also the optimization process itself needs adaptation for the application and optimization of physical triggers as well as a possible downstream robustness analysis.

References

STRUCTURAL OPTIMIZATION METHODS AND TECHNIQUES
FOR ADDITIVE MANUFACTURING

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Abstract

Additive Manufacturing (AM) is a relatively new manufacturing process that can be used to generate complex parts which sometimes conventional processes cannot create. In recent years, users of AM technologies have started to use results from structural and topology optimization techniques to generate better AM designs. However, results from optimization cannot always be successfully or easily printed. In this paper, we describe some methods and techniques that allow the end user to generate structural design proposals which could be manufactured using 3D printing with minimum changes. In general, the methods and techniques described in this work are based on parameterizing the design domain and are developed for gradient-based topology optimization and can optionally be used together with other optimization methods such as shape and sizing. The proposed methods take into consideration irregular FEA meshes commonly used in industrial applications. The main focus of the methods discussed in this paper is to prevent that the final design contains overhang members with shallow angles as such features would either fail or require non-structural supports. The manufacturing requirements are built in the parameterization of the design space and will also be able to impose minimum member size which is also necessary to print 3D printed parts. The methods are discipline independent and have been implemented to be used with responses calculated from different analysis types such as statics, heat transfer, and/or dynamic problems. The discussed methods have been implemented in the GENESIS program and examples, that show their effectiveness, are included.

Key words: Structural Optimization, Additive Manufacturing, Topology Optimization, Overhang Constraints

1. Introduction

Additive Manufacturing (AM) also known as 3D printing is a manufacturing process in which a structure or a part is built with a 3D printing machine in a layer-by-layer fashion. In contrast to other fabrication methods, such as casting or forging, AM has the unique ability to generate complex parts without the need of expensive molds, presses and/or tooling. On the other hand, in contrast to milling, AM does not waste too much material. AM has experienced a rapid growth since the 1980s when important work and first patents were granted. This type of manufacturing technique was first conceived and used to generate prototypes and today is beginning to be used to create final products. The market of AM machine is growing rapidly and the fact that many of the initial patents have expired is allowing new companies to enter the market and reduce the cost of the 3D printers and the material used to print [1].

3D printing technologies according to the ISO/ASTM 52990 Standard (2015) can be categorized in seven processes: a) Material Extrusion; b) Vat Polymerization; c) Powder Bed Fusion; d) Material Jetting; e) Binder Jetting; f) Direct Energy Deposition; g) Sheet Lamination. Details on these processes can be found in references [1] and [2]. Here we will briefly discuss 3 of the most popular processes to give an idea of the variety of technologies available today:
a) Material Extrusion is an AM process in which material is selectively dispensed through a nozzle or orifice. This process uses FFF (Fused Filament Fabrication) technology or FDM (Fused Deposition Modeling). A notable printer manufacturer that uses FDM is Stratasys who has a trademark for FDM. This type of printer makes use of thermoplastics in the form of filament on spools. The type of material to use depends on the quality of results needed. Machine that use material extrusion is found in a wide variety of applications, from personal use to industrial use;
b) Vat Polymerization is an AM process in which a liquid polymer is selectively cured in a vat using a light source. This process uses SLA (Stereolithography) technology. The term SLA was coined by Charles W. Hull, founder of 3D Systems Corporation which is a major producer of 3D printers;
c) Powder Bed Fusion is an AM process in which a thermal energy selectively fuses a region of powder bed. Technologies associated are Selective Laser Sintering (SLS), Direct Metal Laser Sintering (DMLS), Selective Laser Melting (SLM), and Electron Beam Melting (EDM). With these technologies, parts can be made of polymers or metals. A notable producer of SLS machines is EOS from Germany.

Users of AM technologies have started to use results from optimization to generate better AM designs. However, results from optimization cannot always be successfully or easily printed. Brackett et al. [3], in 2011, summarizes the main challenges in topology optimization for AM. In their publication, they describe the challenge of having to generate support structures. Since the time of this publication, several publications have been presented on the subject of self-supporting structures and the use of overhang constraints to avoid the use of non-structural supports. Among the most useful publications applicable to our implementations are of Gaynor [4], Gaynor and Guest [5], and Langelaar [6]. These publications propose filters to the design variable that avoid the creation of overhangs which exceed a critical angle that would require non-structural supports.

2. Overhang Constraints

Overhangs are design features that raise in an angle that deviates from the build directions. Overhangs which is below a certain critical value is undesirable because they may be unstable and might require non-structural supports. These non-structural supports are typically costly, as they require extra material and extra post-processing of the 3D printed parts that contain them. The overhang angles are normally measured from the build plate. The value of the critical overhang angle depends on many factors, among them the 3D process itself and the material used. For the FFF process a typical value is $45^\circ$ [2]. There are several ways to tackle overhang constraints. One way is to create actual constraints that are added to the optimization problem. Another way is to parameterize the design domain so that the creation of overhangs is avoided. We have tried both ways but the latter has produced better results for us.

3. Design Variable Definition

In our work we have used the following design variable parameterization to avoid the growth of shallow overhangs:

$$ Y_j = f(x_j, S_j) $$

(1)

In this parametrization, $x_j$ is an independent design variable measured at a level $j$ of the design space. $Y_j$ is a dependent design variable that is being filtered and used to update the FEA model. $S_j$ is also a dependent design variable that is used to measure the density of the area under the point on space represented by design variable $Y_j$.

The core of the parameterization is to define the function $f$ and the support $S_j$. The function $f$ is defined such a way that the final density (or printed density) should not exceed the density $S_j$ in the supporting region. In Langelaar’s work, the function $f$ is defined as:

$$ f(x_j, S_j) = \text{MIN}(x_j, S_j) $$

(2)

Since the MIN function is not differentiable, Langelaar proposes the use of the following approximation to the MIN function:

$$ \text{SMIN}(x,S) = \frac{1}{2} \times (x+S-((x-S)^+ + \epsilon)^{1/2} + \epsilon^{1/2}) $$

(3)

In the above equation $\epsilon$ is a small number e.g. 0,001 or 0,0001 and its usage allows us to avoid dividing by 0 on sensitivity calculations.

In Gaynor and Guest’s work the function $f$ is defined as:

$$ f(x_j, S_j) = x_j \times S_j $$

(4)

The dependent design variable $S_j$ in Langelaar’s work is defined as:

$$ S_j = \text{MAX}(Y_{j-1}) $$

(5)

In the above equation $\Omega_j$ is a set that contains all variables that can support point $j$. $Y_{j-1}$ are dependent (printed) variables located at a level below level $j$. On a 2D structure, the $\Omega_j$ set usually contains 3 supporting points (at corner and edges it contains the less points). While in 3D structures the $\Omega_j$ set usually has 5 members. Figure 1 shows in the dark 3 $Y_{j-1}$ points. In Figure 1 the built direction is assumed to be vertical, in other words the structure is assumed to be built upward by the 3D printing machine.
As with the MIN function the MAX function is not differentiable, to overcome that Langelaar proposes the use of the following approximation to the MAX function:

$$\text{SMAX}(Y_{j-1}) = \left( \sum_{j} (Y_{j-1})^P \right)^{1/Q} \quad Y_{j-1} \in \Omega$$

(6)

In the above equation $P$ is power that when it grows to infinite would make SMAX converge to MAX. A suggested value is $40, 0. Q = \frac{P + \log(N_s)}{\log(p_0)}$, where $N_s$ is a number of supporting points, and $p_0$ is a number between 0 and 1. A suggested value of $p_0$ is 0.5.

The dependent design variable $S_j$ in Gaynor and Guest’s work is defined as:

$$S_j = H_T(g_k(Y_{j-1})) \quad Y_{j-1} \in \Omega$$

(7)

In the above equation $g_k(Y_{j-1})$ is the average of all points that are in the support region underneath the point $j$. In this case the support region is a conical region situated under the point $j$ and it can be constructed by defining a radial distance $R_j$; $k$ is 1, 2 or larger and depend on $R$. The angle that defines the cone is equal to two times the allowable overhang angle. Figure 2 shows the support region. In this Figure the build direction is assumed upward as in Figure 1.

$$H_T \text{ in Eq. (7) is a thresholding Heaviside function that polarizes the average density to determine whether or not there is enough support for point } j.$$  

$$H_T(g_k) = \left[ \frac{\tanh(Beta_T \cdot T) + \tanh(Beta_T \cdot (g_k - T))}{\tanh(Beta_T \cdot T) + \tanh(Beta_T (1-T))} \right]$$

(8)

In Eq. (8) $Beta_T$ and $T$ are parameters to tune up. As shown in Figure 3, higher values of $Beta_T$ allows the equation to produce sharper polarization (values near 0/1). $T$ is a threshold value that set the amount of density deemed to support adding material.

Figure 3. Thresholding heaviside function for different $Beta_T$ values and for threshold $T=0.1$

Since the dependent design variable $Y_j$ is dependent on the value $S_j$ which in turn is dependent of other values of $Y_{j-1}$ or $Y_{j-1}$ located at lower levels, the evaluation of the design variables has to be performed from the bottom level to the top level. It should be mentioned here that variables on the first (lower) level are assumed to be supported by the build platform, so in that level Eq. (2) is reduced to: $Y_j = x_j$ and $S_j = 1$; if $j = 1$.

4. Implementation

In our work, we have included the support functions and dependent support variables from both Langelaar and
Gaynor and Guest. In our implementation the selection of which support function to use is an option for the user. Our implementation is done in the GENESIS structural optimization software [7] and in such a way, that the overhang constraints can be used with any of the existing responses and can be used simultaneously with other existing optimization types such as sizing, shape, etc.

In GENESIS, the structural optimization problem is solved using the approximation concepts approach [8]. In this approach, an approximate analysis model is created and optimized at each design cycle. The design solution of the approximate optimization is then used to update the finite element model, and a full system analysis is performed to create the next approximate analysis model. The sequence of design cycles continues until the approximate optimum design converges to the actual optimum design. In the mid-seventies Schmit et al. introduced approximation concepts for traditional structural optimization [9-10]. In the late seventies and eighties Starnes et al. [11] and Fleury et al. [12] introduced conservative approximation to improve the approximation used. In the eighties and early nineties, Vanderplaats et al. further improved the quality of approximations by introducing the use of intermediate responses [13-16] also referred as second generation approximations. The approximate problem is solved using either the BIGDOT [17-18] or DOT [19] optimizers. The purpose of using the approximation concepts approach, conservative approximations, and second generation approximations is to reduce the number of design cycles to reduce time. With these approximations, a good answer can be typically found in 10 to 25 design cycles. However, in problems with overhang angle constraint the number of design cycles can grow above 25.

5. Examples

5.1 MBB Example

In this example, the classical MBB beam is designed with and without additive overhang constraint. The optimization is to minimize the overall strain energy (compliance) utilizing no more than 50% of the mass. For additive, the build direction is chosen to be along positive Y direction, and the allowable overhang angle is 45 degrees.

Figure 4. Element densities. On the left without and on the right with overhang constraints

Figure 5. Isosurface densities. On the left without and on the right with overhang constraints
5. 2 MBB Example Results

As shown in Figures 4 and 5, the topology design without overhang constraint forms members with shallow angles. While with overhang constraint, the members are at least at a 45-degree angle as requested. The strain energy for topology without overhang constraint is 2.7578. The strain energy with overhang constraint is 3.0340, which is about 10% higher.

5. 3 Topology Optimization of a 3D Beam

In this example, a cantilever beam is designed with and without overhang angle constraint. The objective in the problem is to minimize the strain energy with a mass fraction constraint of 30%. The build direction is along positive Y direction, and the allowable overhang angle is 45 degrees. The structure is fixed at one side and subject to vertical edge loads at the lower edge of the opposite side.

5. 4 Topology Optimization of a 3D Beam Example Results

As shown in Figures 6 and 7, the results without overhang constraint would need support at the members/surfaces.
with shallow angles. While with the results using overhang constraints, the members are formed with at least 45 degrees as requested.

6. Conclusion

This paper discussed about methods to design structures that can be additively manufactured. The implementation presented allows the user to generate optimal topology design that avoids or reduces the presence of overhang members that would require unnecessary waste of material and expensive trimming. The implementation is general in the sense that it can be used with any response already available in the GENESIS software and it can also be used together with other types of structural optimization types as sizing and shape optimization.

References

A DISCRETE SIZE AND TOPOLOGY OPTIMIZATION METHOD FOR COMPLEX STRUCTURES WITH EXTENDED APPROXIMATION CONCEPTS

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Abstract

This work presents an optimization method with discrete size and topology variables for complex structures, which is extended from an engineering continuous size and topology optimization method and is more useful in the practical engineering design. The method in this study is divided into two stages. In the first stage, with the assumption that size variables can vary continuously in the feasible domain, an optimal solution \( x' \) for the transformed problem can be successfully obtained by the preceding engineering method. Next, by reducing the original list of available cross-sectional dimensions into a sub-space surrounding \( x' \), an ultimate solution selected from the sub-space can be easily determined by using GA with an approximation method. Structural analyses are only conducted following a complete iteration cycle. Numerical examples are established to illustrate feasibility of this method as well as high efficiency, and the optimization results show that the number of structural analyses is dramatically decreased compared with the instances in other articles.

Key words: Discrete Size and Topology Optimization, Approximation Concept, Complex Structure

1. Introduction

Since engineers mostly select the cross-sectional properties from an available list when designing practical structures [1], the discrete size optimization methods have been attracted much attention in the engineering design. Quite a considerable number of these methods are based on GAs, whose effectiveness has been proved with many numerical examples [2]. However, the main drawback of GAs is the tremendous number of structural analyses, which will be implemented every time when the variable values are changed. Except for GAs, a multitude of global optimization methods have been widely utilized to solve the discrete size optimization problems [3]. Nevertheless, due to the limitations of simple constraint or many specific assumptions when solving structural optimization problems, an extensive application of these methods may be impeded.

In recent studies, Dong et al. [4] presented a multi-point approximation method with GA to solve truss topology optimization. Starting from an initial ground structure, a satisfactory topology configuration is established by using GA, and the related cross-sectional areas of truss members can be optimally determined with the dual method. One merit of the method is low computation cost, requiring dozens of structural analyses. The method was developed to tackle the topology optimization problems of complex structures [5]. Nevertheless, this algorithm cannot solve the problem of selecting an optimal solution from a given cross-sectional set.

In this work, an optimization method using GA with extended approximation concepts is constructed to solve the discrete size and topology optimization problems for complex structures. This method involves two main processes. Assuming original problems as an optimization with continuous variables, an intermediate optimal solution \( x' \) can be obtained with the previous method in [5], which will influence the new region for each design variable created in the following process. The region involves only four available dimensions, so it is quite smaller than the original list. Then it is obvious that an ultimate solution can be easily obtained from new variable regions by using a GA with the preceding approximation method. Meanwhile, a multitude of structural analyses is reduced by using the concept of approximation method. An eight-story frame was established to verify the feasibility and efficacy of this strategy, and the results indicate that this method has a highly great capability to solve discrete size and topology optimization for complex structures.
2. Problem Description

In the present study, the optimization model involving two types of variables is established based on a ground structure. Discrete 0/1 variables determine the member in the initial ground structure should be remained or not, and related size variables are created to confirm specific dimensions of the corresponding component. In order not to change the finite element (FE) during the whole optimization procedure, the components removed from the ground structure are set as very small cross-sectional dimensions. The number of the two types of variables can be different, which makes the optimization method have the capability to solve the topology optimization problems of complex structures. The optimization problems can be formulated as follows,

$$\begin{align*}
\text{Find} & \quad X = \{x_1, x_2, \ldots, x_n\}^T \\
& \quad a = \{a_1, a_2, \ldots, a_m\}^T \\
\text{Min} & \quad f(X, a) \\
\text{s. t.} & \quad g_j(X, a) \leq 0, \quad j = 1, \ldots, J_0 \\
& \quad a_i = 0 \text{ or } a_i = 1, \quad i = 1, \ldots, n \\
& \quad x_i \in \{A_i^1, A_i^2, \ldots, A_i^s\}, \quad i = 1, \ldots, n \\
& \quad x_i = x_i^k, \quad k = 1, \ldots, m
\end{align*}$$

(1)

Here, \(X\) is the discrete size variable vector, where \(x_i\) denotes the \(k\)-th size variable and \(n\) is the number of such size variables. \(\{A_i^1, A_i^2, \ldots, A_i^s\}\) is the available list of cross-sectional dimensions for \(x_i\) if the related structural component is retained; \(x_i^k\) is a small value to represent the size variable if its component is not necessary. \(s\) is the number of the available cross-sectional dimensions for the corresponding structural component. \(a\) is the discrete 0/1 variable vector, with \(a_i\) representing the presence(1) or absence(0) of the \(i\)-th structural component. \(a_i^k\) is another representation of \(a_i\) which denotes the \(i\)-th discrete 0/1 variable with respect to the \(k\)-th continuous size variable. \(n\) denotes the total number of the design structural components, \(f(X, a)\) represents the total weight of the structure, \(g_j(X, a)\) represents the \(j\)-th normalized structural constraint response (e.g., displacement or stress constraints), and \(J_0\) is the number of total constraints.

3. Optimization Method

In order to solve the problem in Eq. (1), we present a discrete size and topology optimization method with extended approximate concepts. Firstly, by transforming the primal problem into an optimization problem with continuous size variables, an intermediate optimal solution \(x^*\) can be obtained with the aforementioned engineering method\([5]\). Next, with a narrow sub-space which is a part of the original available set and is surrounded by \(x^*\), the final solution can be easily obtained by using a GA with the preceding approximation concepts. The main process of this method is shown as follows.

3.1 The First-stage Optimization Process

In order to solve the primal problem in Eq. (1), a strategy transforming discrete size variables into continuous ones is utilized in this study. Under this circumstance, the initial problem can be converted into a continuous size optimization problem which is almost identical to the original one except for the feasible domain of \(x_i\). In the new problem, \(x_i\) is defined in a continuous interval \([x_i^L, x_i^U]\), where \(x_i^L\) and \(x_i^U\) are the maximum and minimum of the given set \(\{A_i^1, A_i^2, \ldots, A_i^s\}\), respectively. The transformed problem can be successfully solved by using the aforementioned engineering method, which is operated as follows.

$$\begin{align*}
\text{Min} & \quad f^{(p)}(X, a) \\
\text{s. t.} & \quad g_j^{(p)}(X, a) \leq 0, \quad j = 1, \ldots, J_1 \\
& \quad a_i = 0 \text{ or } a_i = 1, \quad i = 1, \ldots, n \\
& \quad x_i^{(p)} \leq x_i \leq x_i^{(U)}, \quad i = 1, \ldots, n \\
& \quad x_i = x_i^k, \quad k = 1, \ldots, m
\end{align*}$$

(2)
\[ x_{i_{qj}} = \min \{ x_{i_{qj}} , \bar{x}_{i_{qj}} \} \]
\[ x_{u_{qj}} = \max \{ x_{u_{qj}} , \bar{x}_{u_{qj}} \} \]

Here, \( f^{sp}(X, \mathbf{a}) \) and \( g^{sp}(X, \mathbf{a}) \) represent the approximate objective and constraint functions, respectively. \( J_{i} \) is the number of active constraints in the current structural configuration determined by the vector of discrete 0/1 variables \( \mathbf{a} \). \( \bar{x}_{i_{qj}} \), and \( \bar{x}_{u_{qj}} \), denote the move limits of \( x_{i} \) during the \( p \)-th stage.

The approximate functions \( f^{sp}(X, \mathbf{a}) \) and \( g^{sp}(X, \mathbf{a}) \) in Eq. (2) can be both expressed explicitly by using a branched multipoint approximation (BMA) function, which is a weighted Taylor expansion on both branches with different transmutated variable spaces. For further details, please see the previous work[5-6].

2) To solve the first-level approximation problem which still consists of continuous and discrete variables, a layered optimization strategy is constructed[5]. Discrete 0/1 variables are determined with genetic algorithms (GAs) in the external layer, and continuous variables are optimized in the internal layer by establishing the second-level approximation problems. Since the number of constraints is far fewer than that of design variables in general, it is quite efficient to solve the second-level approximation problem by the dual method with small computational costs.

3) A standard GA is used to optimize discrete 0/1 variables in this study. The GA is operated with the simulated roulette wheel selection method as the reproduction operator, as well as crossover and mutation operators[6]. The process of the simulated roulette wheel selection is conducted based on the fitness calculation for each individual in the current population. For each individual in GA, an exterior penalty fitness function involving constraints is constructed based on the preceding approximate concepts, which is detailed in [6].

The procedure of GA will not terminate until the number of evolution generations \( \textbf{G1} \) is larger than the maximum number of generation \( \textbf{MaxG1} \) which is preset.

4) For each member in \( \textbf{GA} \), the discrete 0/1 variables in the first-level approximation problem are fixed as \( \mathbf{a}_{s} \). Under the circumstance, only the size variable corresponding to the retained component should be optimized in the first-level approximate problem. For simplicity, a second-level approximate problem using Taylor expansions and reciprocals with respect to continuous variables is established and in the \( q \)-th step, it can be formulated as follows.

\[
\begin{align*}
\text{find} & \quad \mathbf{X} = \{ \bar{x}_{1}, \bar{x}_{2}, \cdots, \bar{x}_{i} \}^T \\
\text{min} & \quad \tilde{f}^{sp}(\tilde{X}, \mathbf{a}_{s}) = f^{sp}(X_{qj}, \mathbf{a}_{s}) + \sum_{i=1}^{J_{2}} \frac{\partial f^{sp}(\tilde{X}, \mathbf{a}_{s})}{\partial x_{i}} (\bar{x}_{i} - \tilde{x}_{i_{qj}}) \\
\text{s. t.} & \quad \tilde{g}^{sp}(\tilde{X}, \mathbf{a}_{s}) = g^{sp}(X_{qj}, \mathbf{a}_{s}) - \sum_{i=1}^{J_{2}} x_{i_{qj}} \frac{\partial g^{sp}(X_{qj}, \mathbf{a}_{s})}{\partial x_{i}} (\frac{1}{x_{i}} - \frac{1}{x_{i_{qj}}}) \leq 0 \quad j = 1, \cdots, J_{2}
\end{align*}
\]

Where \( \tilde{X} \) denotes the vector of remained continuous size variables and \( I \) is the number of these variables. \( \tilde{f}^{sp}(\tilde{X}, \mathbf{a}_{s}) \) and \( \tilde{g}^{sp}(\tilde{X}, \mathbf{a}_{s}) \) are the approximate objective and constraint functions at the \( q \)-th step, respectively, while \( J_{2} \) represents the number of retained constraints. \( x_{i_{qj}} \) and \( \bar{x}_{i_{qj}} \) are the upper and lower bound of \( \bar{x}_{i} \) at the \( q \)-th step, and \( \tilde{x}_{i_{qj}} \) as well as \( \bar{x}_{i_{qj}} \) denote the move limit of \( x_{i} \) in the \( k \)-th step. Then with a dual method, the optimal solution can be easily obtained in the second-level approximation problem.

3.2 The Second-Stage Optimization Process

As the solution obtained above cannot be utilized directly since it is just an intermediate value, a second-stage optimization strategy is established. This strategy is presented to solve the discrete size optimization with fixed topology configuration determined in section 3.1, which can be formulated as follows.
\[
\begin{aligned}
\text{find } & \quad X = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_I)^T \\
\text{min } & \quad f(X) \\
\text{s.t. } & \quad g_j(X) \leq 0 \quad j = 1, \cdots, J_z \\
& \quad x_k \in \{\text{SA}_k, \text{SA}_k^+, \text{SA}_k^+, \text{SA}_k^+\} \quad k = 1, \cdots, I
\end{aligned}
\]

where \(X\) is the vector of size variables which is eventually in the first-stage optimization process, with \(I\) denoting the number of such variables, \(f(X)\) and \(g_j(X)\) represent the objective and constraint functions, respectively, and \(J_z\) is the number of retained constraints. \(\{\text{SA}_k, \text{SA}_k^+, \text{SA}_k^+, \text{SA}_k^+\}\) represents the \(k\)-th current available list involving four members which are selected from the original set of cross-sectional dimensions. These members are adjacent to the corresponding variable \(x_k\) of the optimal solution \(x^*\) obtained from section 3.1.

This problem can be successfully solved by the following strategy:

1) Due to the implicit property of constraints mentioned in section 3.1, the preceding first-level approximation method is utilized to express the constraints explicitly in Eq. (4). In the \(p\)-th step, the approximate problem can be formulated as follows.

\[
\begin{aligned}
\text{find } & \quad \tilde{X} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_I)^T \\
\text{min } & \quad f^{(p)}(\tilde{X}) \\
\text{s.t. } & \quad g^{(p)}_j(\tilde{X}) \leq 0 \quad j = 1, \cdots, J_z \\
& \quad x_k \in \{\text{SA}_k, \text{SA}_k^+, \text{SA}_k^+, \text{SA}_k^+\} \quad k = 1, \cdots, I
\end{aligned}
\]

Here, \(f^{(p)}(X)\) and \(g^{(p)}_j(X)\), are the approximate objective constraint functions which can be explicitly expressed by using the BMA function. The problem in Eq. (5) is an optimization problem with only discrete size variables and can be easily tackled by a standard GA.

2) The standard GA is similar as that mentioned in section 3.1, except for the method of coding variables. Each dimension in the given list in Eq. (5) is replaced by a corresponding integer value ranged from one to four. The real values of cross-sectional dimensions are only used in the fitness calculation which is identical to that in section 3.1. When the fitness calculation is completed, a simulated roulette wheel selection method is used as the reproduction operator, followed by crossover and mutation operations. The procedure of GA will stop when the number of evolution generations \(G^2\) is larger than the maximum number of pre-set generation \(\text{MaxG}^2\).

4. Numerical Example

4.1 15-Bar Planar Truss

A 15-bar planar truss structure is established to illustrate the feasibility and efficiency of this method for solving the discrete size and topology optimization problems in multiple loading cases, and it is a typical instance researched by Wang with a sequential two-level method[8]. The ground structure and its primary physical dimensions are shown as Figure 1, with two corner points fixed in this structure. Two different loading cases are listed as Table 1. The material elasticity modulus is \(6,897 \times 10^6\) N/cm\(^2\), and the density is 0.002768 kg/cm\(^3\).

![Figure 1. Ground structure of 15-bar planar truss](image-url)
Table 1. Loading Cases for the 15-Bar Planar Truss

<table>
<thead>
<tr>
<th>Case</th>
<th>Node</th>
<th>( P_x )</th>
<th>( P_y )</th>
<th>( P_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>-445000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>-445000</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>-445000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-445000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>-445000</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>-445000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The design requirements of this structure are as follows: the allowable stresses for all bars are \( \pm 17243.5 \text{ N/cm}^2 \), and the vertical displacement limit for node 5 is no more than 2.032 cm. The bar members are sorted into eight groups which are corresponding to eight dependent size variables and are listed as Table 2. The available cross-sectional areas for bar members are 6.452, 9.677, 22.581, 32.258, 45.161, 70.968, 83.871, 103.226, 129.032, 161.29, 193.548 cm\(^2\).

Table 2. Groups for All Bar Members

<table>
<thead>
<tr>
<th>Group</th>
<th>Bar Member</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-4, 2-8</td>
</tr>
<tr>
<td>2</td>
<td>4-6, 6-8</td>
</tr>
<tr>
<td>3</td>
<td>3-4, 7-8</td>
</tr>
<tr>
<td>4</td>
<td>3-6, 6-7</td>
</tr>
<tr>
<td>5</td>
<td>4-5, 5-8</td>
</tr>
<tr>
<td>6</td>
<td>1-3, 2-7</td>
</tr>
<tr>
<td>7</td>
<td>3-5, 5-7</td>
</tr>
<tr>
<td>8</td>
<td>5-6</td>
</tr>
</tbody>
</table>

All the initial values of dependent variables are given as 70.968 cm\(^2\), with their upper and lower bounds set as 6.452 and 193.548 cm\(^2\), respectively. The GA parameters in this example are the same as the two examples above. Table 3 lists the results obtained with the sequential two-level method and the proposed method.

Table 3. Optimization Results for 15-Bar Planar Truss

<table>
<thead>
<tr>
<th>Var (unit: cm(^2))</th>
<th>Previous work(^{(1)})</th>
<th>This Study</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>70,968</td>
<td>70,968</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>32.258</td>
<td>32.258</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>32.258</td>
<td>6.452</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>6.452</td>
<td>6.452</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>6.452</td>
<td>6.452</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>6.452</td>
<td>6.452</td>
</tr>
<tr>
<td>( x_8 )</td>
<td>45.161</td>
<td>45.161</td>
</tr>
<tr>
<td>Max displacement, cm</td>
<td>1.975(^*)</td>
<td>1.975(^*)</td>
</tr>
<tr>
<td>Max stress, N/cm(^2)</td>
<td>15424(^*)</td>
<td>15424(^*)</td>
</tr>
<tr>
<td>Weight, kg</td>
<td>396.66</td>
<td>396.66(^*)</td>
</tr>
<tr>
<td>No. of S. A</td>
<td>—</td>
<td>17</td>
</tr>
</tbody>
</table>

\* : results calculated with MSC Patran/Nastran

From the results in the table, it can be seen that with a quite small number of structural analyses, the optimal
weight obtained in this work is the same as that in the previous work, which illustrates high efficiency and effectiveness of the proposed method. The configuration and the iteration process of structural mass are shown in Figure 2 and Figure 3, respectively. It is noteworthy that the structural mass keeps stable at a few iterations and then decreases in the following process, which demonstrates that the proposed method has some capability of escaping a locally optimal solution.

Figure 2. Optimization configuration in this study

Figure 3. Iteration process of weights for 15-bar planar truss

4.2 Eight-story Frame

To verify the feasibility of the present method for discrete size optimizations of frame structures, an eight-story frame is established, whose details are explained in the works by Schenvels[7]. The bar members are categorized into 8 groups in [7], and each group must select a section from the AISC database of W-shapes whose profile is shown in Figure 4.

Figure 4. W-shapes in the AISC database

There is only a single constraint; the lateral displacement of the top left node must be no more than 2 in. However, by using the optimal solution obtained in [7] to reconstruct the eight-story frame with MSC, Patran/Nastran, the displacement of the top left node can reach even 37.59 in. Under this circumstance, a new constraint which expands the original allowable displacements to 37.6 in is established in this study. And for simplicity, twenty W-shapes selected from the original AISC database are chosen in the available list, consisting of W18×40, W18×46, W16×31, W16×26, W18×35, W14×22, W18×71, W18×65, W18×60, W16×100, W16×89, W16×36, W14×283, W14×211, W14×145, W14×109, W14×90, W14×68, W14×38, W12×152.

One difficulty of this example is that each cross-section should determine four relevant dimensions simultaneously in the first-stage optimization problem, which directly influences the establishment of the sub-space in the second-stage optimization. In this study, the dimensions whose cross-sectional moment of inertia is close to that of x' are selected.
And since the objective and constraint functions are convex if their partial derivatives for all size variables are positive, a new size variable $H_L$ is utilized to represent the concave variable $H_L$, which is written as $H_L = \frac{(H_2 - H_1)}{2}$. The details of the size variables for all W-shapes are listed as Table 4.

<table>
<thead>
<tr>
<th>Var (unit: in.)</th>
<th>initial value</th>
<th>lower bound</th>
<th>upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>H2</td>
<td>13.74</td>
<td>13.71</td>
<td>18.47</td>
</tr>
<tr>
<td>Ht</td>
<td>0.335</td>
<td>0.335</td>
<td>2.07</td>
</tr>
<tr>
<td>t</td>
<td>0.23</td>
<td>0.23</td>
<td>1.29</td>
</tr>
<tr>
<td>W</td>
<td>4.77</td>
<td>4.77</td>
<td>14.82</td>
</tr>
</tbody>
</table>

The GA parameters in this example are set as follows: the population size $Popsize = 50$, the maximum generation number $MaxG1 = MaxG2 = 50$, crossover probability $Pc = 0.9$, and mutation probability $Pm = 0.05$. The details of optimization results, contrasted with the previous work, are listed in Table 5. And the iteration process of frame weight is presented in Figure 5.

<table>
<thead>
<tr>
<th>Group</th>
<th>Results in [7]</th>
<th>This study</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>W18×40</td>
<td>W16×31</td>
</tr>
<tr>
<td>2</td>
<td>W16×31</td>
<td>W16×31</td>
</tr>
<tr>
<td>3</td>
<td>W16×26</td>
<td>W14×22</td>
</tr>
<tr>
<td>4</td>
<td>W12×16</td>
<td>W14×22</td>
</tr>
<tr>
<td>5</td>
<td>W18×35</td>
<td>W18×35</td>
</tr>
<tr>
<td>6</td>
<td>W18×35</td>
<td>W18×35</td>
</tr>
<tr>
<td>7</td>
<td>W16×31</td>
<td>W16×31</td>
</tr>
<tr>
<td>8</td>
<td>W14×22</td>
<td>W14×22</td>
</tr>
</tbody>
</table>

Max displacement, in $37.59^°/2^°^*$, $37.41^°$

Weight $6872.4^*'/6984.2^°^*$, $6522.61^°^*$

No of S, A, -- 15

* $^*$ : results calculated with the Patran/Nastran
* $^*$ : results obtained in [7]

From the results above, it can be seen that the optimum weight in this study is less than the result in [7]. In addition, the number of structural analyses is quite small in this paper, which illustrates the high efficiency of the two-level approximation approach.
5. Conclusion

An optimization method with discrete size and topology variables is developed in this study based on the previous optimization method. Two stage procedures are implemented in this method. Assuming the original size variables are continuous ones firstly, the transformed problem can be directly solved with the previous method by constructing the first-level approximation problem and tackling it with a layered optimization strategy. Then by regarding the solution optimized from the preceding optimization process as an initial point, the ultimate solution can be successfully obtained with a pure genetic algorithm in the second-stage optimization problem based on the aforementioned multi-point approximation method. Because the approximation concepts are employed to establish the response functions instead of numerous structural analyses, the computational cost is significantly reduced. The optimization method is tested with a typical numerical example composed of frame structures and compared with the previous work, great effectiveness of this approach is verified as well as high efficiency.

Reference

SOLVING TOPOLOGY OPTIMIZATION WITH \{0,1\} DESIGN VARIABLES AND MATHEMATICAL PROGRAMMING: THE TOBS METHOD

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Abstract

Gradient-based topology optimization problems using binary \{0,1\} design variables have been considered extremely challenging or even impossible to be solved, leading to the creation of relaxed problems such as the ones used in the popular density-based methods. This work investigates a combination of numerical ingredients that allows the original \{0,1\} problem to be solved sequentially and with formal mathematical programming. This method was named Topology Optimization of Binary Structures (TOBS) and it was developed in Sivapuram and Picelli [1] and Sivapuram et al. [2]. In the TOBS method, the objective and constraint functions are sequentially linearized using Taylor’s first order approximation. Sensitivity numerical filtering is applied to avoid checkerboard and mesh-dependency issues. Integer Linear Programming (ILP) is used to compute the optimized solutions for the linearized problems at each iteration, allowing the method to accommodate the set of constraints explicitly. The constraint targets are relaxed to allow only small changes in topology during an update and to ensure the existence of feasible solutions for the ILP. This work discusses the possible advantages and disadvantages of the TOBS method, as well as its details and comparisons with other discrete methods.

Key words: Structural Optimization, Topology Optimization, Discrete Methods, Binary Variables, Integer Programming

1. Introduction

Topology optimization for elastic structures is essentially a material distribution problem with binary \{0,1\} design variables, where 0 means the absence of material (void) and 1 represents the location of solid material. The binary problem has been considered for decades very hard or even impossible to be solved using formal mathematical programming. Hence, the idea of relaxing the binary constraint by allowing intermediate densities became popular with the Solid Isotropic Material with Penalization (SIMP) formulation [3]. The SIMP method usually starts with intermediate (gray scale) densities and during optimization it steers the solution to a nearly solid/void design. The advent of projection schemes helped the SIMP solution to converge to a solid/void design [4].

The first attempt to carry out topology optimization of continuum structures using binary variables was made by Xie and Steven [5]. The idea is that inefficient material can be removed from the structure using sensitivity information. This method was called Evolutionary Structural Optimization (ESO). The method became popular after convergent and mesh independent solutions were presented by Huang and Xie [6], who proposed a Bi-directional ESO (BESO). The BESO method is based on a heuristic design update scheme that allows material to be also added to the structure. In the BESO algorithm, the structure is usually (but not necessarily) first considered as a full solid design and a target volume is used to quantify the amount of removed/add material until convergence, only allowing \{0,1\} variables. One advantage of using binary variables is in the solution of problems where the structure interacts with other physics, e. g., fluid-structure interaction design, where the switch between solids and fluids/voids is straightforward and the equilibrium conditions at the interfaces are explicit [7]. On the other hand, the main disadvantage of the BESO method is that its update scheme is based on a target volume, therefore always requiring the presence of a volume constraint in the optimization formulation. This precludes problems such as mass minimization or multiple constraints to be solved in a schematic way.

Beckers [8] introduced formal mathematical programming to solve a structural topology optimization problem with...
binary \{0, 1\} variables, Beckers’ method was based on sequential approximate programming and a Lagrangian dual method optimizer. In this way, the solutions obtained were convergent and mesh independent but limited to very few constraints. Svanberg and Werme \cite{9} found that if only one element is binarily changed (from material to void or from void to material), the new global stiffness matrix is just a low-rank modification of the old one, therefore validating discrete sensitivities. Relying on that, Svanberg and Werme \cite{10} introduced integer linear programming to solve a sequentially approximate problem with binary variables. As a drawback, their choice of hierarchically refining the finite element mesh can lead to mesh dependent solutions. To the best of our knowledge, we presented the first convergent and mesh independent solutions for topology optimization with binary design variables and integer programming in Sivapuram and Picelli \cite{1}. The method so called Topology Optimization of Binary Structures (TOBS) combines four numerical ingredients; sequential problem linearization, constraints’ relaxation (move limits), sensitivity filtering and an integer programming solver. This represents a more general scheme for topology optimization with \{0, 1\} variables than \cite{8} and \cite{10}, as also showed by Sivapuram et al. \cite{2} for microstructural optimization problems with multiple nonvolume constraints. Munk \cite{11} used the same four numerical ingredients as the TOBS method in \cite{1} and developed some heuristics for the move limits. However, in our understanding, the author in \cite{11} is misleading when calling his method BESO because the intrinsic heuristic BESO update scheme is not present anymore. In fact, the TOBS method applies well known numerical ingredients previously used by other methods in order to find an algorithm that leads to convergent binary solutions in a general topology optimization framework. Very recently, Liang and Cheng \cite{12} proposed a similar method for binary topology optimization including a few other techniques. Liang and Cheng applied the Canonical Dual Theory \cite{13} to solve the integer problem while the TOBS method uses a branch and bound solver \cite{14} which worked well for a wide range of problems investigated.

This paper aims to provide details of the TOBS method as well as brief comparisons with the BESO method \cite{8} and the methods by Svanberg and Werme \cite{10} and Liang and Cheng \cite{12}. The remainder of the paper is as follows, Section 2 describes the TOBS method and section 3 summarizes it in steps. Section 4 presents numerical results and section 5 concludes the paper.

2. The TOBS Method

2.1 Problem Linearization

A generic topology optimization problem with objective function \( f(x) \), constraints \( g_i(x) \leq \bar{g}_i \), where \( x \) is the design variables, is herein linearized via Taylor’s first order approximation. The approximate problem is given by:

\[
\begin{align*}
\text{minimize} & \quad \frac{\partial f}{\partial x} \Delta x^i \\
\text{subject to} & \quad \frac{\partial g_i}{\partial x} \Delta x^i \leq \Delta g_i^i, \quad i \in [1, N_c] \\
& \quad \Delta x^i \in \{-x^i, 1-x^i\}, \quad j \in [1, n_d]
\end{align*}
\]

(1)

where \( N_c \) is the number of constraints and \( N_d \) is the number of design variables, \( \Delta g_i^j \) is the change in the \( i \)th constraint function at the \( k \)th iteration and \( \Delta x^d \) represents the change in design variables. At the end of each iteration, the design variables are updated as \( x^{k+1} = x^k + \Delta x^d \).

In order to maintain the validity of the linear approximations and the truncation error low, an extra constraint is added:

\[
\| \Delta x^d \|_1 \leq \beta N_d
\]

(2)

This equation implies that the number of flips between solid to void and vice versa is constrained to be a fraction \( \beta \) of the total number of design variables. By limiting the change of the structural topology at every iteration, the number of solid material layers to be removed is also kept low.

2.2 Move Limits

Gradient-based optimization requires an initial solution to start with, which might lie in the infeasible space of the optimization problem. To guarantee the linearized constraints are satisfied at every iteration, the constraints are relaxed, i.e., the values \( \Delta g_i^j \) are chosen such that the linearized problem has a feasible solution. The relaxation for a
constraint $g$ at the iteration $k$ is given by:

$$
\Delta g^k = \begin{cases} 
-\varepsilon_i g^k & \text{if } g^k < (1-\varepsilon_i) g^k \\
\varepsilon_i g^k & \text{if } g^k \in [(1-\varepsilon_i) g^k , (1+\varepsilon_i) g^k] \\
-g^k & \text{if } g^k > (1+\varepsilon_i) g^k
\end{cases}
$$

(3)

where $\varepsilon_i$ and $\varepsilon_e$ are small numbers chosen to keep $\Delta g^k$ small enough so that the subproblem in Eq. (1) has a feasible solution. Throughout this work, $\varepsilon_i = \varepsilon_e = \varepsilon$ is used.

2.3 Sensitivity Analysis and Filtering

The TOBS is a gradient-based topology optimization method. The first gradients (sensitivities) must be computed in order to evaluate the linearized functions from Eq. (1) and solve the subproblem at iteration $k$. The sensitivities can be evaluated using any sensitivity analysis method [15], e. g., analytical, semi-analytical or finite differences if only the corresponding binary values are used. In this work, we have used the semi-analytical method to evaluate the sensitivities of the mean compliance $C$ of an elastic structure, which final function can be written as:

$$
\frac{\partial f(x)}{\partial \varepsilon_j} \sim \frac{\Delta C}{\Delta \varepsilon_j} = -\frac{1}{2} u_j^T K_j^{-1} u_j,
$$

(4)

where $u_j$ and $K_j$ are, respectively, the vector of displacements and the stiffness matrix of the $j$th element. Using the same method, the sensitivity of the volume with respect to each design variable is the volume of the element.

Numerical filtering on the sensitivity field has been massively used in structural topology optimization to avoid numerical issues, e. g., mesh dependency and checkerboard problems, besides guaranteeing a sort of length scale. Herein, we filter sensitivities by first creating a nodal sensitivity field. In a regular grid, the nodal sensitivity (with respect to a virtual nodal design variable $y$) is the average of the sensitivities of each element the node is connected to. The filtered sensitivity field with respect to the design variables is reconstructed from the nodal sensitivity field by using weighted-distance averaging in a neighborhood of each finite element. This filtered sensitivity is given by:

$$
\frac{\partial f(x)}{\partial \varepsilon_j} = \sum_{m \in N} w_m \frac{\partial f(x)}{\partial y_m} \frac{\partial y_m}{\partial \varepsilon_j}
$$

(5)

where $N$ is the set of nodes in a neighborhood of the $j$th element. We use a circular neighborhood of radius $r$ with the center being the centroid of the $j$th finite element.

We have observed small oscillations in the structural members due to the high number of local minima for some thermomechanical problems. Therefore, we use the stabilization scheme proposed by Huang and Xie [6], which is the averaging of the sensitivity field with its previous iteration. It can be pointed out that this stabilization scheme is not essentially required but its effects improve convergence considerably, as also showed in [6]. We have also observed that normalizing sensitivities helps the optimizer to find an integer solution faster since all sensitivity numbers will have a similar range, which also avoids ill-conditioning. This only implies that the subproblem in the $k^{th}$ iteration is scaled.

2.4 Integer Programming

The linear subproblems are solved using Integer Linear Programming (ILP). An ILP problem is the same as Linear Programming (LP) problem with additional constraints that the feasible space of design variables contains only integers. ILP problems can be solved using branch-and-bound methods by initially solving the LP problem, i. e., without the integer constraints on design variables, and the obtained solution is used to create branches of LP problems with additional inequality constraints to achieve integer solutions [14]. In this work, the package CPLEX is used for solving the ILP problems. The cplexmip function of the CPLEX MATLAB package uses the branch-and-bound method for solving mixed integer linear problems.

3. Code and Algorithm

A demonstration code with a possible implementation of the TOBS method is available upon request by e-mail at rpicelli@usp.br. This code is based on the following steps.

1. Discretize the design domain, apply loads and boundary conditions.
2. Choose the TOBS parameters, $\varepsilon$ to relax each constraint function, $\beta$ to limit the total amount of solid-to-void and void-to-solid flips, filter radius $r$ and convergence criteria.

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3. Perform Finite Element Analysis (FEA) of the structure.
4. Compute sensitivities and the objective and constraint functions’ values.
5. Filter the sensitivities of each of the function.
6. Average the sensitivity fields with the corresponding fields from previous iteration.
7. Normalize the sensitivities.
8. Build the optimization problem (Eq. (1)) with relaxed constraints (Eq. (3)) and add the extra flip limit constraint (Eq. (2)).
9. Solve the problem with ILP.
10. Update design variables $x^{k+1} = x^k + \Delta x^k$.
11. If converged, stop. Otherwise, return to step 3.

4. Discussions and Numerical Results

Although Beckers [8] has brilliantly obtained mesh independent and convergent solutions in 1999, the role of using binary design variables in topology optimization of continuum structures have been mostly carried out by the BESO method. In its early years, BESO has been criticized by its lack of formal mathematical programming. Nevertheless, it became a popular algorithm due to its efficiency and the ability to produce binary solutions that are comparable with other methods. However, the problems that BESO can solve are within the range that include only volumetric constraints because of BESO’s target volume update. Therefore, the TOBS method can replace the BESO method for more complex problems out of the BESO range, like mass minimization and multi-constrained problems. The convergence of the nonvolume constraints is better in TOBS because they are dealt explicitly, i.e., without using any Lagrange multipliers like BESO. The advantages of using binary variables include mostly problems where material interpolation (like used in density-based methods) is challenging or leads to numerical issues. Example of these problems are thermoelastic design, fluid-structure interaction, acoustic-structure interaction, i.e., design-dependent physics. For single physics problems, density-based methods outperform discrete methods because they present a higher design space and have more powerful optimizers available. Furthermore, density-based methods have proven to solve a wide range of challenging problems. Therefore, future recommendations for discrete methods include new applications and the development of new optimizers that are faster and can solve large scale problems.

We advocate that the four numerical ingredients from section 2 are the recipe to carry out topology optimization with binary variables. Naturally, there are options between the available tools for each of the ingredients, e.g., different integer programming solvers like cplexmilp from CPLEX library or intlinprog from MATLAB. In order to illustrate this recipe, we start comparing the TOBS with the method by Beckers [8]. For the benchmark MBB beam example with a regular unitary 150x50 mesh, Young’s modulus 1.0, Poisson’s ratio 0.3 and unitary applied force, Figure 1 shows the solutions for minimum structural compliance with a volume fraction of 30%. Herein, $\epsilon = 0.01$ and $\beta = 0.05$. Beckers’ method applied all the four numerical ingredients and it was the first to obtain mesh independent solutions. The main difference between the TOBS and Beckers’ [8] method lies in the integer program solver, in which Beckers applied a Lagrangian dual solver, limiting the application to a few constraints. The TOBS method uses integer linear programming, able to accommodate a higher number of constraints but slower in obtaining the solution. Another difference is the introduction of a perimeter constraint in [8], not present in the TOBS solution from Figure 1(b), which possibly explains the different number of holes in the final topology. The final compliance was 189.72 for the TOBS method and 193.75 for Beckers’ optimized solution.

![Figure 1. Solutions obtained for (a) MBB beam with the (b) TOBS method and (c) Beckers’ [8] method.](image)

The work by Svanberg and Werme [10] seems to indicate the authors did not use sensitivity filtering. Instead, they proposed a mesh refinement scheme in which the optimization problem is solved for each mesh level. Although they also used constraint limits (relaxation), mesh refinement guaranteed the validity of the discrete sensitivities, as very
few element layers are removed every time. The TOBS method guarantees these move limits with constraint relaxation (parameter $\varepsilon$) and the extra constraint on the binary flips (parameter $\beta$). Furthermore, sensitivity filtering plays the crucial role of extrapolating the sensitivity field so as void elements nearby high stressed regions can return to solid. Figure 2 shows the solutions for an L-beam structure for minimum compliance and volume constraints with the method by Svanberg and Werne [10] and the TOBS, using the same initial parameters as the previous example.

![L-beam example](image)

Figure 2. Solutions obtained for (a) L-beam example with the (b-c) method by Svanberg and Werne [10] with mesh refinement and (d-i) the TOBS method.

Liang and Cheng [12] recently proposed the use of Canonical Dual Theory (CDT) with the argument that the canonical relaxation solver can accommodate more design variables than integer programming. Nevertheless, it seems that the number of constraints solved by CDT is limited, similarly to Beckers [8]. Figure 3 presents the solutions of a regular 300x150 mesh structural design for minimum compliance with 50% of constrained volume fraction using the TOBS method (with same parameters of the previous examples), our own implementation of the BESO method and the solution presented by Liang and Cheng [12]. The specific particularities of these methods include that Liang and Cheng [12] use move limits based on the volume or densities while the TOBS method relaxes (limits) the constraint changes at every iteration. The BESO method does not use either sequential approximate problems or formal mathematical programming and its move limit is actually based on a heuristic update scheme. Anyhow, the three solutions showed to be similar local minima. Liang and Cheng [13] also presented a similar solution but using quadratic programming, which decreased the number of iterations by half.

![MBB beam](image)

Figure 3. Solutions obtained for (a) MBB beam with (b) the TOBS method, (c) our own implementation of the BESO method [8] and (d) sequential linear integer programming via canonical relaxation by Liang and Cheng [12]. Total of 45000 design variables.

With the previous comparisons, one can observe that Beckers [13], Liang and Cheng [17], BESO [8] and TOBS methods present similar solutions when solving a minimum compliance with volume constraints problem. Beckers [13] and Liang and Cheng [17] did not show volume minimization examples, although their methods are theoretically able to solve them. We would like to point out that one good use of the TOBS method is for solving minimum volume problems with multiple constraints as showed by Sivapuram et al. [2] and by Munk [16].

5. Conclusions

This paper formalizes the four numerical ingredients of doing topology optimization with binary $\{0, 1\}$ design variables, specifically in the context of the proposed TOBS method. Numerical results show that the method presents comparable solutions with literature (except for Svanberg and Werne [10]) when compared to a simple compliance


minimization with volume constraint problem.

Acknowledgments

Renato Picelli would like to thank the support of São Paulo Research Foundation (FAPESP), grants 2018/05797-8 and 2019/01685-3.

References

OPTIMAL ARCHGRIDS REVISITED: VARIATIONAL APPROACH AND NUMERICAL METHODS

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Abstract

This paper addresses the Rozvany-Prager problem of finding minimum weight structures that are entirely in compression, or in tension, while transferring a transmissible load to pinned supports. Optimal designs, termed “Prager-structures”, are composed of arch-like members that are bending and shear-free. The topology of such forms reflects the type of external loading; optimal structures are sparse for concentrated loads, and dense for distributed ones. The framework of this study is two-dimensional and focused on archgrids with the rectangular pattern of members. The computational part of the research is based on the variational approach to the Rozvany-Prager problem in the continuous and discrete settings.

Key words: Optimal Archgrids, Prager-Structures, Minimum Weight, Variational Approach

1. Introduction

In the 1970s, G. I. N. Rozvany and W. Prager initiated a study on optimal design of bar-like continua subjected to forces that migrate in space along fixed lines of action [1]. The position of load depends on the shape of the structure, and vice versa; hence migrating forces are sometimes termed “design dependent load” [2]. However, Rozvany-Prager framework accommodates gravity, self-weight, or snow, but not wind or fluid pressure. More generally, in the Cartesian coordinate system OXYZ, forces are allowed to shift, without rotation, along lines OZ, perpendicular to the plane OXY. Therefore, the phrase “transmissible load” [3], is perhaps more adequate here. The goal in Rozvany-Prager problem is to find the minimum weight structure that is entirely in compression, or in tension, while carrying the transmissible load between pinned supports. Most of the contribution at the early-stage of investigation in this field of structural optimization is due to G. I. N. Rozvany and his collaborators, cf. [4] for complete list of references. Nevertheless, to recognize the important role of W. Prager in originating the research, optimum forms are named “Prager-structures” [5].

For a load acting over given one-dimensional segment, Prager-structure takes the form of a funicular, which is an idealized bending- and shear-free arch whose neutral axis forms a plane curve. By intuition, as well as mathematical reasoning, this curve is continuous for any load, which makes the topology of one-dimensional Prager-structures steady. In two-dimensional case, however, shapes and topologies of optimal forms strongly depend on loading. Prager-structures spanned over two-dimensional domains are composed of bending- and shear-free members whose neutral axes form a single surface, not necessarily continuous. In certain load cases, e. g. point loads, optimal structures materialize as systems of arch-like spatial curves [5-6]. Mathematically speaking, transmissible loads represented by functions with two-dimensional support of zero Lebesgue measure are optimally carried by sparse systems, while dense systems are optimal for loads represented by functions whose support is of non-zero Lebesgue measure.

The interest in Rozvany-Prager theory had dropped in the 1990s and has been revived only recently by posing the optimization problem in two, mutually dual, variational forms, cf. [6-7]. The primal formulation, \( P \), involves maximization of the functional \( z \rightarrow f(z) \), where \( z : \Omega \rightarrow R \) is a scalar field representing elevation function of the load carrying system, and \( \Omega \) denotes a given, two-dimensional domain. The dual approach, \( P^* \), includes minimization of dual functional, \( Q \rightarrow f^*(Q) \). Here, \( Q : \Omega \rightarrow R^2 \) denotes a vector field of certain static quantity; it is introduced in the sequel.
In the current research, we use this novel variational perspective in the computational shaping of Prager-structures. Our specific goal is to address the Rozvany-Prager problem by three methods. Method (1) carries out the analysis of \((P)\) in the continuous setting, with trial functions decomposed in a certain functional basis. Method (2) deals with \((P^*)\) in similar manner. Method (3) involves the discrete setting of both variational problems. All methods use optimization procedures coded in Mathematica and Matlab. Partial results of the research have been announced in [8-9].

Prager-structures are close siblings of Michell structures. However, Michell structures can resist both tensile and compressive stresses, while Prager-structure members are by assumption subjected to stresses of the same sign. We refer the Reader to [10] for striking demonstration of the difference between Michell’s and Prager’s notion of optimality. Further discussion of this topic lies outside of the scope of this paper. More complete elaboration can be found in [6, section 6.2]. Alternative methods of construction of grid-shells are discussed in [11].

2. Optimal Rectangular Archgrids

Fix a two-dimensional, convex domain \(\Omega\) with Lipschitz boundary \(\partial\Omega\) on the Cartesian plane \(OXY\), and let the transmissible load \(q : \Omega \to \mathbb{R}\) act in the direction \(OZ\), perpendicular to \(\Omega\). Assume that \(\Omega\) is covered by densely packed line segments arranged in two collections; the first includes lines parallel to \(OX\), and the second one those parallel to \(OY\). Both collections from a beam-like continuum \(C\) simply supported at \(\partial\Omega\).

Next, consider two domes made from arches whose chords coincide with straight-line segments from \(C\). Neutral surfaces of domes coincide; they are given by a single elevation function \(z : \Omega \to \mathbb{R}\), see [7] for the proof of this fact. By assuming that beams in \(C\) run along \(OX\) and \(OY\), we limit the discussion to archgrids whose chords cross each other to form a series of rectangles; hence the term “rectangular archgrids”, see Figure 1 for graphical explanation of symbols used in this discussion. Removing the limitation imposed on the pattern of beams in \(C\) is possible in both theoretical and computational capacity. We briefly refer to this generalized approach in Example 4.1.

Optimal elevation function \(z_{opt} : \Omega \to \mathbb{R}\) is calculated through the following (primal) variational problem,

\[
\begin{align*}
(P) \quad & Z_{opt} = \max \{ \int_{\Omega} q z \, dx \, dy \mid z_{\partial\Omega} = 0, \frac{1}{L_1(y)} \int_{\Omega} \left( \frac{\partial^2 z}{\partial x^2} \right)^2 \, dx \leq 1, \frac{1}{L_2(x)} \int_{\Omega} \left( \frac{\partial^2 z}{\partial y^2} \right)^2 \, dy \leq 1, \text{ in } \Omega \}.
\end{align*}
\]

Inequalities in \((P)\), termed “mean squared slope conditions”, or “Prager conditions”, after [1], are sharp. They are satisfied as equalities along certain lines \(x = \text{const}\), and \(y = \text{const}\); these lines determine the subdomain \(\Omega_{opt} \subseteq \Omega\). Optimal archgrid roofs \(\Omega_{opt}\) and the part \(\Omega \setminus \Omega_{opt}\) remains uncovered.

![Figure 1](image-url). Explanation of symbols used in the variational formulation of the optimal rectangular archgrid problem.
Assume now that the beam-like continuum $C$ is loaded by $q$, and let $Q: \Omega \to \mathbb{R}^2$ represent a shear force vector field that is in static equilibrium with $q$. All such fields constitute the set
\[
\sum_s \phi_s(\Omega) = \left\{ Q \bigg| \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} + q = 0 \quad \text{in} \quad \Omega \right\}
\] (1)

Variational problem dual to (P) reads
\[
(P^*) \left| Z_s = \min \left\{ \int_{\Omega} \sqrt{\rho_s(Q_{1y}(x,y), Q_{2y}(x,y))} \, dx + \int_{\Omega} \sqrt{\rho_s(Q_{1x}(x,y), Q_{2x}(y,y))} \, dy \bigg| Q \in \sum_s(\Omega) \right\}
\]
where
\[
\rho_s(x,y) = \left( \int_{\chi^{(c)}} (f(x,y)) \, dx \right)^{1/2}, \rho_s(x,y) = \left( \int_{\chi^{(c)}} (f(x,y)) \, dy \right)^{1/2}
\] (2)
The duality gap is zero, i.e., $Z_s = Z_s$, cf. [7], and it turns out that the volume of optimal archgrid is given by
\[
V_{vol} = \frac{2}{\sigma_c} Z_s = \frac{2}{\sigma_c} \left| q_{vol} \right| \, dx \, dy
\] (3)
where $\sigma_c$ stands for the allowable compressive stress.

3. The Numerical Approach to Archgrid Optimization

3.1 The Continuous Setting

Variational problems in section 2 have the mathematical structure of the Monge-Kantorovich optimal transport problem. Hence, minimizers in (P*) would be measures instead of functions, and this fact needs to be appropriately manifested in computational techniques for both (P) and (P*). However, these techniques are not fully developed yet, and for this reason, the computational approach used in this paper is classical. Namely, we choose the Hilbert space $H^1(\Omega; \mathbb{R})$ for the space of trial functions. This space is separable; hence it contains a countable set $B$, such that solutions to (P) and (P*) are sought in the linear span of $B$. In this research, we choose the Fourier basis of trigonometric func-
tions for $B$. Next, we proceed with the Ritz method to find approximate solutions in the finite dimensional subset $B_s$ of $B$. To facilitate the exposition of the topic, in the remainder of the current section we restrict our considerations to rectangular domains $\Omega = [-L_x, L_x] \times [-L_y, L_y]$, see [9] for more complete discussion of this issue.

Write $f: \Omega \to \mathbb{R}$ for a trial function in either (P) or (P*). Due to the rectangular geometry of $\Omega$, it is convenient to set
\[
f(x,y) = \sum_{i=1}^n \sum_{j=1}^n f_{ij} \varphi_i(x) \varphi_j(y)
\] (4)
where the products of functions $\varphi_i, \varphi_j, i = 1, \ldots, n, \ j = 1, \ldots, n,$ constitute the set $B_s \subset B$, $n \in \mathbb{Z}^+$. Both variational formulations impose particular conditions on trial functions. Therefore, in (P) we require
\[
z(x,y) = \sum_{i=1}^n \sum_{j=1}^n z_{ij} \sin(a_i x) \sin(b_j y)
\] (5)
with $a_i = i \pi / (2 L_x)$, $b_j = j \pi / (2 L_y)$, $\tilde{x} = x + L_x$, and $y = y + L_y$. In (P*), in turn, we fix
\[
Q_1(x,y) = \sum_{i=1}^n \sum_{j=1}^n Q_{1ij} \cos(a_i \tilde{x}) \sin(b_j \tilde{y}), \quad Q_2(x,y) = \sum_{i=1}^n \sum_{j=1}^n Q_{2ij} \sin(a_i \tilde{x}) \cos(b_j \tilde{y})
\] (6)
\[
q(x,y) = \sum_{i=1}^n \sum_{j=1}^n q_{ij} \sin(a_i \tilde{x}) \sin(b_j \tilde{y})
\]
Differential constraints in (1) restrict the coefficients $Q_1$ and $Q_2$ in (6), to such that $\beta \dot{Q}_1 + a \dot{Q}_2 = q_0$

3.2 The Discrete Setting

In the discrete setting, beam-like continuum $C$ is made up of the finite number of beams along $OX$ and $OY$. Similarly, loads are represented by a finite number of concentrated forces acting along lines parallel to $OZ$. Consequently, (P) naturally reduces to a problem with the finite number of unknown elevations at beam intersection points. Each variable represents the position of a force along the line of its action and, consequently, the elevation point of an
archgrid. The problem \( P^* \) is dealt with in similar manner. In what follows we denote the number of beams along \( OX \) and \( OY \) by \( k_x \) and \( k_y \).

4. **Examples**

4.1 **Square \( \Omega, \ L_x=L_y=L, \) with \( q=\text{const} \) in the entire domain**

Load function and shape of \( \Omega \) are symmetric with respect to \( OX \) and \( OY \). For this, Prager’s conditions in \( (P) \) and equilibrium equations in \( (P^*) \) are considered along one axis only and we set \( n_x = n_y = n \). For \( 15 \leq n \leq 100 \), the volume ranges from \( V_{15} = 3.6722qL^3/\alpha \) to \( V_{100} = 3.6813qL^3/\alpha \).

![Figure 2. Archgrid in Example 4.1 obtained through \( (P) \) in the continuous setting for \( n=75 \). Volume \( V=3.6812qL^3/\alpha \) and maximum elevation at \( (x, y) = (0, 0) \) equals \( z_{\max} = 0.8275L \).](image)

![Figure 3. Archgrid in Example 4.1 obtained through generalized \( (P) \) in the discrete setting for \( k_x = k_y = 80 \). Volume \( V=3.5410qL^3/\alpha \) and maximum elevation at \( (x, y) = (0, 0) \) equals \( z_{\max} = 0.8305L \).](image)

![Figure 4. Michell structure in Example 4.1. Volume \( V=3.5002qL^3/\alpha \). The elevation at \( (x, y) = (0, 0) \) varies between \( 0.77L \) to \( 0.85L \).](image)
Cartoon in Figure 2 shows the archgrid obtained through (P) in the continuous setting with \( n_1 = n_2 = n = 75 \), see also [9]. The Numerical duality gap between solutions to (P) and \((P^*)\) is less than 0.02\% for \( n_1 = n_2 = n \), and \( 15 \leq n \leq 75 \) is assumed in either approach, primal or dual. Plot in Figure 3 shows the solution to the generalized problem (P) in the discrete setting with the ground structure of arches parallel to \( OX \), or to \( OY \), or running at \( \pm 45 \) deg to these directions. Figure 4 presents the Michell archgrid resisting both tension, up to given yield stress \( \sigma_t \), and compression. However, tension in optimal archgrid members was numerically suppressed by assuming \( \sigma_t / \sigma_c = 0.1 \). In the latter approach, the ground structure consists of bars running in all directions; hence the volume of Michell structure is the lowest among all solutions to (P) in Example 4.1. However, numerical procedure for the Michell problem splits the transmissible load among several archgrids. Consequently, optimal elevation function \( z_{opt} = z_{opt}(x, y) \) is not determined in this case, see Figure 4.

**4.2 Square \( \Omega \) with \( q = \) const Over the Central Part of the Domain**

Consider now a square domain from the previous example and the constant transmissible load acting above its central part \( \Omega = [-0.5L, 0.5L] \times [-0.5L, 0.5L] \). In this example, the volume calculated for \( 15 \leq n \leq 100 \), \( n = n_1 = n_2 \), ranges from \( V_{15} = 1.5602qL^2 / \sigma_c \) to \( V_{100} = 1.5607qL^2 / \sigma_c \). Plot in Figure 5 shows the archgrid obtained through (P) in the continuous setting with \( n_1 = n_2 = n = 75 \), cf. [9]. Its volume \( V = 1.5607qL^2 / \sigma_c \). The numerical duality gap between solutions to (P) and \((P^*)\) is less than 0.09\% for all \( n \geq 15 \).

![Figure 5. Archgrid in Example 4.2 obtained through (P) in the continuous setting for \( n=75 \).
Volume \( V=1.5607qL^2 / \sigma_c \). Red dashed lines indicate non-optimal arches cut out from the original grid.](image)

**4.3 Cross-shaped Domain**

A cross-shaped domain discussed in this example is built from five \( L \)-by-\( L \) square subdomains. Rectangular archgrid obtained through (P) in the discrete setting with \( k_x = k_y = k = 151 \) is displayed in Figure 6.

**5. Discussion of Results**

Variational approach to the Rozvany-Prager problem produces accurate results. The numerical duality gap, not fully discussed in this paper, is close to zero, which confirms the reliability of the numerical scheme. Approximate solutions differ by less than 0.01\% for \( n_1 = n_2 \geq 15 \) in Example 4.1 and for \( n_1 = n_2 \geq 30 \) in Example 4.2.

The algorithm is robust. It is capable of making clear-cut distinction between optimal parts of the structure and nonoptimal ones, i.e. those that do not carry any load; hence are redundant from the engineer’s point of view.

Computational algorithm is also CPU time efficient. The waiting period in the continuous setting increases significantly with \( n_1 , n_2 (k_x, k_y \text{ in the discrete one}) \), but this fact is of lesser importance in light of the accuracy of the method.

Archgrid surfaces shown in Figures 2 and 5 are not convex, despite each arch is subject to compression only. Let us
stress that even the non-convex plane arches are tension-free; the normal stress is uniform in the entire structure and equal to $-\sigma$. By enlarging the arch-like ground structure, thus allowing also for bars running at ±45 deg to OX and OY directions, the surface enveloping the family of arches, cf. Figure 3, is convex. Michell-like structure in Figure 4 is also convex.

![Archgrid in Example 4.3](image)

**Figure 6.** Archgrid in Example 4.3 obtained through $(P)$ in the discrete setting for $k=151$.

**Acknowledgments**

The paper was prepared within the Statutory Research Grant realized at the Faculty of Civil Engineering, Warsaw University of Technology.

**References**

TOPOLOGY OPTIMIZATION OF TRUSS AND CONTINUUM STRUCTURES UNDER NON-STATIONARY STOCHASTIC EXCITATIONS

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Abstract

The topology optimization of truss and continuum structures under non-stationary stochastic band-limited white noise excitations are studied in the paper. The optimization model of truss and continuum structures are formulated by taking variances of displacements minimization as objective with volume as constraints. The explicit time domain method (ETDM) is applied to solve the stochastic responses and the sensitivities with respect to the design variables. With sensitivity analysis, the nodal positions and the cross-sectional areas of truss structures are optimized simultaneously by using the globally convergent method of moving asymptotes (GCMMA). Unlike the topology optimization for truss structures, the Adjoint Variable Method (AVM) is used to establish the explicit expression of the sensitivity of the stochastic responses for continuum structures. The element pseudo-densities of continuum structures are renewed by optimal criteria (OC) method to obtain the optimal topological shape of the continuum structures. Finally, two numerical examples illustrate the accuracy of the proposed method.

Key words: Topology Optimization, Truss and Continuum Structures, ETDM, Stochastic Excitations

1. Introduction

Traditional approaches to design structures depend extraordinarily on the designers’ prior experience and some common rules. Topology optimization methods have been intensively studied in recent decades. The topology optimization methods can not only help to save investment cost, but also lead to significant improvements in the performance of the structures. Thus far, most of the research work has focused on the topology optimization of structures under static or deterministic dynamic excitations. Recently, topology optimization subject to random excitations has attracted more and more attention due to the practical engineering requirements for complex working conditions, such as seismic hazards and wind damages. For example, the topology optimization for continuum structures [1-4] and for outriggers applied in structures [5] under stationary random excitations are studied. In order to meet the actual working conditions, the layout optimization of multi-component structures considered both static loads and white noise stochastic loads is investigated [6]. In the quasi application of civil engineering, a framework for optimal stiffness distribution design under non-stationary seismic and stationary wind loads is developed in [7].

Generally in the topology optimization of dynamic problems of multi-degree of freedom structural systems, there are three different sets of methods, which namely the time domain, frequency domain, and state space methods, can be used to solve the dynamic responses and sensitivity analysis [5]. But for non-stationary processes problems, in which the stochastic statistical properties are time-variant, the time domain is a priority method compared with the others. However, the computational cost of the time domain method is quite expensive. Therefore, in this study, the explicit time domain method (ETDM) is implemented which can greatly improve the efficiency of the traditional time domain method in solving non-stationary processes problems.

The ETDM has been proposed in solving random responses of the dynamic system under stationary or non-stationary stochastic excitations [8-9]. For topology optimization, the explicit time domain sensitivity analysis method of structural responses under non-stationary random excitations is also proposed by using direct differentiation method and adjoint variable method [10] and are applied in the optimization of the lateral bracing system of frame structures [11]. In this work, topology optimization of truss and continuum structures under non-stationary stochastic band-limited white noise excitations are studied based on ETDM.
2. ETDM of Structural Response Under Stochastic Excitation

The response analysis approach with ETDM for multi-DOF truss structures under stochastic excitations is introduced. The dynamic equation subjected to a stochastic excitation can be written as:

$$M\ddot{U} + C\dot{U} + KU = LX(t)$$  \hspace{1cm} (1)

where $M$, $C$ and $K$ are mass, damping and stiffness matrices of the structure, respectively; $\ddot{U}$, $\dot{U}$ and $U$ are acceleration, velocity and displacement vectors, respectively; $L$ is the location vector of the excitation, $X(t)$ is a non-stationary stochastic excitation.

Generally, only some of the key points’ certain responses are needed for the calculation. By introducing ETDM, the historical displacement response of the $n$th node at moment $i\Delta t$ is

$$u_i^n = \mathbf{\varphi} V_i = \mathbf{\varphi}^T B \mathbf{\dot{R}},$$ \hspace{1cm} (2)

where $\mathbf{\varphi}$ is the location vector of the displacement component of the $n$th node in the structure. If the stochastic excitation is a stochastic process, the variance of the random response of the $n$th node at moment $i\Delta t$ can be expressed as follows,

$$\text{var}(u_i^n) = \mathbf{\varphi}^T B \text{cov}(\mathbf{\dot{R}}, \mathbf{\dot{R}}) (\mathbf{\varphi}^T B)^T$$  \hspace{1cm} (3)

The covariance $\text{cov}(\dot{\mathbf{R}}, \dot{\mathbf{R}})$ of the random excitations can be calculated by the correlation function of the random excitations. To efficiently obtain the coefficient matrix $B$, it is necessary to carry out one unit triangle pulse historical analysis. The specific deduction process can be found in reference [13].

3. Dynamic Topology Optimization Formulation

3.1 Topology Optimization Model of Truss Structures

To reduce vibration and save material, a stochastic dynamic optimization model of truss structures under non-stationary stochastic excitation is stated as follows,

$$\begin{align*}
\text{min; max} &: \text{var}(u_i^n) & (n = 1, 2, \cdots, p) \\
\text{subject} &: & & & V = \sum_{e=1}^{m} A_e l_e \\
&: & & & A_\mathbf{L} \leq A_e \leq A_\mathbf{U} \\
&: & & & \mathbf{\bar{X}} \leq \mathbf{X}_i \leq \mathbf{\bar{X}}_i, \mathbf{\bar{Y}} \leq \mathbf{Y}_i \leq \mathbf{\bar{Y}} \\
&: & & & u_i^n = \mathbf{\varphi}^T V_i = \mathbf{\varphi}^T B \mathbf{\dot{R}}
\end{align*}$$  \hspace{1cm} (4)

Where $V$ is the total volume of the truss structure. $A_e$ denotes the sectional area of the $e$th bar. $l_e$ denotes the length of the $e$th bar. $\text{var}(u_i^n)$ denotes the displacement variance of the $n$th node at time $t$. The lower and upper bounds of the sectional area of the bars are $A_\mathbf{L}$ and $A_\mathbf{U}$. $\mathbf{X}_i$ and $\mathbf{Y}_i$ are node coordinates of the $n$th node varied from $\langle \mathbf{X}, \mathbf{Y} \rangle$ to $\langle \mathbf{\bar{X}}, \mathbf{\bar{Y}} \rangle$. $p$ and $m$ are the number of nodes and bars respectively.

3.2 Topology Optimization Model of Continuum Structures

The density interpolation scheme of material is adopted to construct the relation of the relative density of the continuum structure element and the material properties as follows [14]:

$$\begin{align*}
\mathbf{m}_e &= x_m \mathbf{m}_a \\
\mathbf{k}_e &= \frac{15x^3_e + x_e}{16} \mathbf{k}_a
\end{align*}$$  \hspace{1cm} (5)

where $\mathbf{k}_e$ and $\mathbf{m}_e$ are the stiffness matrix and mass matrix of the element $e$, $\mathbf{k}_a$ and $\mathbf{m}_a$ denote the mass matrix and stiffness matrix when the element is solid. $x$ denotes the relative density of element $e$; $N_e$ is the total number of the elements. To reduce vibration and save material, the optimization model of continuum structures is formulated as follows.
\[ \text{find: } 0 < x_{\min} \leq x_i \leq 1 \quad (e = 1, 2, \ldots, N_e) \]
\[ \min: \max_{\tau} (\text{var}(u^{\tau})) \quad (n = 1, 2, \ldots, s) \]
\[ \text{s. t.: } \sum_{i=1}^{n} x_i V_{n_i}^{\tau} \leq V^* \]
\[ u^{\tau} = \phi B R \]

Where \( x_{\min} \) is the lower bound of the set of design variables defined by element pseudo-density. Here, \( x_{\min} = 0.001 \) is used to prevent the mass, stiffness and damping matrices from becoming singular, \( \max(\text{var}(u^{\tau})) \) is the maximal displacement variance of the \( n \)th degree of freedom during the time \( T \); \( V^* \) is the upper bound of the displacement variance; \( s \) denotes the total number of the constraints, \( x_i \) and \( N_i \) have the same meaning in Eq. (10); \( V_{n_i}^{\tau} \) is the initial volume of the element \( e \). \( \phi' \) and \( u' \) have the same meaning in Eq. (4).

3. 3 Sensitivity Analysis of Truss Structures

The sensitivity of the responses with respect to the design variable \( \theta \) can be deduced as follows,
\[
\begin{align*}
M \dddot{U}_{i} + C \ddot{U}_{i} + K \dot{U}_{i} &= - \frac{\partial K}{\partial \theta} U_{i} - \frac{\partial C}{\partial \theta} \ddot{U}_{i} - \frac{\partial M}{\partial \theta} \dddot{U}_{i},
\end{align*}
\]

(7)

Where \( \frac{\partial U_i}{\partial \theta} \), \( \frac{\partial \dot{U}_i}{\partial \theta} \) and \( \frac{\partial \ddot{U}_i}{\partial \theta} \) are the sensitivities of displacement, velocity and acceleration with respect to the design variable \( \theta \) at moment \( i \Delta t \), respectively. Similar to solving Eq. (3), the time explicit expression of the sensitivity of the stochastic response variance with respect to the design variable \( \theta \) at moment \( i \Delta t \) can be calculated as follows.
\[
\frac{\partial \text{var}(u^{\tau})}{\partial \theta} = 2(\phi' B_{\tau}) \text{cov} (\dot{R}_i, \dot{R}_j) (\phi' B_{\tau})^T
\]

(8)

Where \( \phi' \) is the same meaning in Eq. (7). Due to space limitation, the specific deduction process of Formula (12) can be found in reference [13].

3. 4 Sensitivity Analysis of Continuum Structures

For the sensitivity analysis problem with much more design variables and less constraint functions, the computational efficiency of AVM is significantly higher than that of the direct method [15]. For a dynamic problem, when a response of a certain degree of freedom at the predefined time \( t_n \) is concerned, the response can be written in the following form.
\[
\begin{align*}
u_{i,n} &= \int_0^T \phi_{i}(t) \cdot U(t) \, dt
\end{align*}
\]

(9)

To calculate the sensitivity of the response \( \nu_{i,n} \) with respect to a design variable \( \theta \), an arbitrary adjoint vector \( \lambda \) is introduced and the augmented response \( L(u_{i,n}, \lambda) \) is defined as follows:
\[
L(u_{i,n}, \lambda) = u_{i,n} + \int_0^T \lambda(t) \left( M \dddot{U} + C \ddot{U} + K \dot{U} - L X \right) dt
\]

(10)

As Eq. (1) is always true, the derivatives of \( u_{i,n} \) and \( L(u_{i,n}, \lambda) \) with respect to \( \theta \) must be identical, which means,
\[
\frac{\partial L(u_{i,n}, \lambda)}{\partial \theta} = \frac{\partial u_{i,n}}{\partial \theta}
\]

(11)

According to the deduction process of the AVM [16], the time-domain explicit expressions of the sensitivity of the maximal displacement variance of one degree of freedom during the time \( T \) has the following form.
\[
\frac{\partial \text{cov}(u_{i,n})}{\partial \theta} = 2(\phi' C_{\tau}) \text{cov} (\dot{R}_i, \dot{R}_j) (\phi' B_{\tau})^T
\]

(12)

3. 5 Optimization Solver

Based on the sensitivity analysis, the optimal solution of truss structures can be solved by GCMMA [17]. The optimization model of topology optimization for continuum structures belongs to a single objective and single constraint, the OC method is used as the optimizer due to its numerical simplicity and numerical efficiency [18].
4. Numerical Examples

Example 1: Topology Optimization of 2D Truss Structure

Let consider a 12-bar truss structure of the size shown in Figure 1. The elasticity modulus of the material is $E = 1 \times 10^8$ Pa. The density of the material is $\rho = 2600$ kg/m$^3$. The height and width of the initial truss are 1 meter and 2 meters, respectively. The damping model is the Rayleigh damping and the first two order damping ratio is 0.05. This truss structure is subjected to a non-stationary limited white noise excitation $F(t)$, which can be simulated by $F(t) = g(t)f(t)$. $g(t)$ is the uniform modulation function. The expression is $g(t) = e^{i - (\omega + \eta)/\omega_t^2}$ and $q_1 = T/2$, $q_2 = T/5$. $T$ is the duration of the excitation. $f(t)$ is the band-limited white noise with zeros mean. The power spectrum is $S_f(\omega) = S_0$ if $|\omega| < \omega_c$, else, $S_f(\omega) = 0$. The spectrum intensity $S_0 = 1 \times 10^4$ N$^2$/(rad/s). The optimization objective is to minimize the maximal horizontal displacement variance of the point $A$ at the bottom of the right side, and the constraints total volume is 7 cubic meters. The duration of the excitation is 1 second and the time step is $\Delta t = 1/(\omega_c/(2\pi))/5$. The optimization results are shown in Figure 2 with different excitations. The convergence curve of the volume constraint with iteration steps are shown in Figure 3. The convergence curve of vertical standard deviation of displacement of point $A$ with iteration steps are shown in Figure 4. The comparison of the results of the optimized structures with different excitations is listed in Table 1.

![Figure 1. The Initial Structure of Truss](image)

![Figure 5. Initial Design Model of 2D Plate Structure](image)

![Figure 2. The Topology Structures of Trusses with Different Excitation](image)

!(a) Static Load (b) $\omega_b = 30$ rad/s (c) $\omega_b = 90$ rad/s (d) $\omega_b = 120$ rad/s.

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Example2: Topology Optimization of 2D Continuum Structure

The plane plate structure is defined as a rectangular domain of size $1\text{m} \times 3\text{m} \times 0.05\text{m}$. It is clamped at the left side as shown in Figure 5. The elasticity modulus of the material is $E=2\times10^6\text{ Pa}$. The density of the material is $\rho=5800\text{ kg/m}^3$. The Poisson ratio is 0.3. Here, the domain is meshed into $40\times120$ Q1 elements. The Rayleigh damping is considered and the first two damping ratios of the initial structure are assumed fixed at 5%. A non-stationary limited white noise stochastic excitation $F(t)$ is applied in the middle of the right side. The excitation has the same form as example 1.

In this example, the volume fraction is 0.3. The duration of the excitation is 1 second and the time step is $\Delta t = 1/(\omega_s/(2\pi))/5$. The optimization results are shown in Figure 6 with different excitations. The convergence curve is not listed due to space limitation. The comparison of the results of the optimized structures with different excitation is listed in Table 2.

![Figure 3. The convergence curve of volume constraint](image)

![Figure 4. The convergence curve of standard deviation](image)

![Figure 6. The topology structures of continuum structures with different excitation](image)

(a) Static load (b) $w_s=100\text{rad/s}$ (c) $w_s=200\text{rad/s}$ (d) $w_s=300\text{rad/s}$.

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</table>

Table 2. Comparison of Optimization Results Under Static Load, Stationary and Non-Stationary Random Excitation

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Observing the structures in Figure 2 and Figure 6, the optimization result is very close to the static optimization result when the cut-off frequency of band-limited white noise is lower than the first-order frequency of the static optimization structure. When the cut-off frequency of band-limited white noise is higher than the first-order frequency of the static optimization structure, the optimization results are obviously different with the static optimization result. From the
5. Conclusion

In this paper, the topology optimization of truss and continuum structures under non-stationary stochastic band-limited white noise excitations are studied. Because the structural optimization is a dynamic iteration process and sensitivity analysis is needed on each step, the computational efficiency of the sensitivity analysis is of crucial importance. Accordingly, the explicit time domain sensitivity analysis method based on unit pulse response analysis can analyze the stochastic response sensitivity of the objective function and constraint functions with respect to the design variables with high efficiency. Then, combined with GCMMA and OC algorithm, it is effective in dealing with the topology optimization of truss structures and continuum structures under non-stationary stochastic excitations. The optimization results of the two examples illustrate that the standard deviation displacement responses of the structure are reduced and the goal of saving material is achieved. Moreover, the sensitivity analysis and optimization method can be applied to the topology optimization for truss and continuum structures under non-stationary seismic excitation; this will be the focus of the next work.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 11002056, 11372004 and 51878252).

References

IMPOSING MANUFACTURING CONSTRAINTS IN TOPOLOGICAL OPTIMIZATION OF 2D FUEL CELL FLOW PROBLEMS USING OPENFOAM

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Abstract

Proton-Exchange Membrane Fuel Cells (PEMFC) are systems that directly convert chemicals into electricity by means of an electro-chemical reaction between hydrogen and oxygen. Following the concerns related to climate change and the search for cleaner, safer and more efficient power sources, Hydrogen PEMFC is a promising option. Indeed, PEMFC has a silent operation, high reliability, low working temperature and they only produce water as waste. Nevertheless, to rival the Internal Combustion Engine (ICE), PEMFC shall decrease their manufacturing cost, increase their lifetime and their efficiency as well as improve the thermal management, among others. To achieve this goal, a simulation environment has been developed to couple OpenFOAM (to simulate the fluid flow) with mathematical programming Optimization Algorithms, where length scale manufacturing constraints can be imposed to propose new designs of the gas channel network for the bipolar plates of the fuel cell.

Key words: Fluid Flow Topology Optimization, Manufacturing Constraints, PEMFC, OpenFOAM

1. Introduction

Topology optimization answers the question posed in engineering, where to put material within a prescribed design domain in order to obtain the best performance. Using the homogenization theory, the seminal work [1] established a framework to solve mechanical problems. Later on, the method was popularized with the SIMP method [2-3], the level set method [4], among others to account for more complex physical problems. The interested reader can refer to the exhaustive review presented in [5].

Topology optimization has been widely used to solve problems in structural mechanics [6], optics [7], acoustics [8], fluid mechanics [9] among others.

The seminal work for the topology optimization of fluids [9] present a density-based approach to minimize the dissipated power in a Stokes flow. It has then been used multiple times to study Stokes flows under steady state conditions [10-12] or the more complex incompressible Navier Stokes equations using FEMLAB (nowadays COMSOL) [13-15]. Unsteady flows have been also studied [16-17] while some researchs have been done considering the utilization of the level set method [18-19].

Other physical problems have also been addressed, like the minimization of the thermal compliance for incompressible Navier Stokes fluids in laminar steady state [20-24], which has a mathematical approach close to the one used in minimum compliance.

Electrochemical reactions and transport problems have been studied in [25-27], mainly by means of COMSOL software tool coupled with Matlab, considering the complexity of the physical phenomena.

Finally, other cost functions have been considered, such as maximization of the diodicity [28], maximization of the mixing performance [29], maximization of the generation rate of vanadium species [30] or maximization of the secondary flow [31]. It has to be noted that all of these studies were based on an academic objective rather than for a practical application.
2. Description of the Optimization Problem

The objective is to define the channel network layout of the bipolar plate of a fuel cell using Fluid Flow Topology Optimization (FFTO) subject to design constraints. To achieve this goal, a simulation environment has been developed to couple OpenFOAM (to simulate the fluid flow) with an environment including Optimization Algorithms, with the addition of length scale manufacturing constraints.

Firstly, we solve the unconstrained minimization problem of the total pressure loss Eq. (1) using the reference solver available in OpenFOAM, called adjoint Shape OptimizationFOAM, which is based on the theoretical developments of [32] and [33]. This solver simulates the incompressible Navier-Stokes equations in steady-state condition Eq. (2) and Eq. (3) and can be considered an evolution of the mathematical framework proposed in [9]. In here, it was considered the simulation of the Stokes flow, where the equations are combined with Darcy's law by means of a Brinkman penalization, which results in a density-based method to perform the topology optimization.

The optimization problem can be stated as:

\[ \min J = \int_{\Gamma} \left( \rho \frac{1}{2} \frac{1}{2} \right) \]  
\[ (R_1, R_2, R_3)^T = (v \cdot \nabla) v + \nabla p - \nabla \cdot (v(v + (v)^T)) + \alpha v \]  
\[ R_i = -\nabla \cdot v \]  

Considering \( J \) as the objective function, \( \Gamma \) is the boundary of the design domain, \( v \) is the velocity field of the flow, \( p \) is the pressure, \( \rho \) is the density, \( v \) the kinematic viscosity of the working fluid and \( \alpha \) is the local inverse permeability, which is the design variable in the problem (0 in the fluids zones and \( a_{\text{max}} \) in the porous/solid zone).

A particular feature of this solver, is the fact that it uses the steepest descent method as the optimization algorithm, which doesn’t allow the consideration of constraints into the problem. As a reference, Figure 1 shows the topology obtained for the optimization problem defined in Eqs. (1-3).

![Figure 1. Optimized topology for the Minimization of the total pressure loss—no constraints](image)

3. Optimization Algorithm Modification and Addition of a Volume Constraint

As stated before, in order to further improve the OpenFOAM implementation, a simulation environment was created in order to first replace the steepest descent algorithm by the Method of Moving Asymptotes (MMA) [34] and second implement a well know volume constraint into the optimization problem. Figure 2 shows the optimized topology with a 50% volume constraint.

As expected, the solver tries to design a nozzle while forcing the apparition of material in order to reach the imposed volume constraint. It is worth to highlight the fact that, while in structural mechanics problems such as the compliance minimization, we start with a design respecting the volume constraint and then try to redistribute the material into an optimized way, in fluid mechanics we start from a fully fluid domain in order to get, at first, a fully developed flow and then try to force the apparition of material in order to fulfill the volume constraint.
4. Incorporation of Length Scale Manufacturing Constraints

The next step in the evolution of the methodology, which aims at designing the gas channel network of a bipolar plate working on a fuel cell, is to consider length scale manufacturing constraints, such as maximum channel size and minimum gap separation of the channels.

The reference work on the field of maximum size constraints \[35\] was latter on complemented by several research. In this work, we adapt in particular the constraint presented in \[36\], where a definition of the maximum size constraint was presented considering an efficient aggregation strategy to speed up the process and improve its convergence, to deal with fluid flows.

In our case, the local maximum size constraint can be defined as:

\[
\text{g}^m = \epsilon - \sum_{\Omega} \frac{V_k \theta}{V_{\Omega}} a_{\text{max}} \leq 0
\]  (4)

where \(\epsilon\) is a small positive number to relax the problem, \(\Omega\) is the maximum size region of the element \(e\), \(V_k\) is the element volume, \(V_{\Omega}\) is the volume of the maximum size region and \(a_{\text{max}}\) represent the solid fraction of the element. The constraint restricts the amount of material (void) around the element \(e\) by imposing at least \(\epsilon\) amount of void (material) in the element.

Figure 5 shows the test domains to control the maximum size and the minimum gap (to be explained after) and Figure 3 shows the results obtained for different values of the maximum size region \(R_{\text{max}}\). The interested reader can check the respective article to get more details into the derivation of the constraint \[36\].

Analyzing Figure 3 we can see that the maximum size constraint, effectively divide the main channel into smaller ones respecting the constraint. Nonetheless, there is no control on the size of the separation wall.

The next step is to define a minimum gap constraint, in order to impose a minimum separation between channels. Following the theoretical developments proposed in \[37\], it can be stated that the minimum gap constraint can be defined in the same way than the maximum size constraint of Eq. (4), where instead of using \(\epsilon\) in the definition of the constraint, we will use \(\epsilon_2\) defined in Eqs. (5-7). The minimum gap constraint finally reads as expressed in Eq. (8),
with all the parameters definitions being the same as in [37].

Figure 4 shows the results for different values of the maximum size region \( h \).

\[
\begin{align*}
\theta_i &= 2\alpha \cos \left( 1 - \frac{2r_{\text{max}}}{r_i} \right) \\
\theta_s &= 2\alpha \cos \left( 1 - \frac{2(r_s - r_{\text{min}})}{r_s} \right) \\
\epsilon_\theta &= \frac{\theta_s - \theta_i + \sin \theta_i - \sin \theta_s}{2\pi} \\
\epsilon_s &= \sum \gamma_i v_i \frac{a}{a_{\text{max}}} \leq 0
\end{align*}
\]

It can be noted that the constraint is taken into account by the optimizer, but nevertheless, a few convergence issues arise and may be addressed in the future developments. The two most important are the apparition of channels that or linked only to the outlet and not to the inlet, which is allowed by the fact of being working with a porous media (Figure 4-b) and the presence of channels open to the boundaries (Figure 4-c), mainly due to the fact that the design domain should be extended after the physical boundaries, in order to avoid geometrical instabilities related to the filters or constraints in the boundary.

Figure 5. Test domains to control the minimum gap (left) and the maximum size (right)

### 5. Conclusions

A simulation environment has been designed to couple OpenFOAM (to simulate the fluid flow) with the optimization algorithm MMA [34], in order to expand the actual capabilities of OpenFOAM related to the field of Topology Optimization, allowing it to consider different sets of constraints, like minimum/maximum volume or manufacturing constraints like maximum size and minimum gap.
Firstly, the minimization of the total pressure loss was solved considering a predefined volume constraint on the amount of solid material into the design domain. Secondly, manufacturing constraints were added to the simulation environment, starting with an adaptation of the maximum size constraint to be applied to the case of fluid flows. Afterwards, the minimum gap constraint was added to the optimization environment in order the better reflect the nature of the bipolar plates design, mainly by imposing a minimum wall dimension to separate the flow channels.

Finally, it was shown in an academic test case that it is possible to perform a topology optimization of a fluid flow, considering at the same time volume and manufacturing constraints, which opens the door to simulate real life engineering cases, like for example the design of bipolar plates.

The next step of this research is to consider the electrochemical nature related to the fuel cells operation by means of the addition into the simulation of the hydrogen consumption. Also, a new objective function will be implemented in order to perform a topology optimization based on the electrochemical aspects of the fuel cells. In parallel, the convergence problems witnessed in the implementation of the minimum gap constraint will be addressed.

Acknowledgments

This research is supported by the project INOXYPEM: Prototypage de plaques bipolaires en acier revêtu pour piles à combustible PEM. Programmation FEDER 2014-2020. Funded by Fonds structurels et d’investissement européens and Region Wallone de Belgique.

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BIDIRECTIONAL EVOLUTIONARY STRUCTURAL OPTIMIZATION WITH RANDOM STARTS

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Abstract

The original and current versions of bidirectional evolutionary structural optimization start with a solid pre-defined domain and they evolve into the best topology as a stopping criterion is met. In such approaches, the results are always identical if input parameters are the same. Our proposal is to start the optimization process with a random density between 0 (void) and 1 (solid) in the pre-defined domain instead of the fully solid density. Therefore, the search will be started with different patterns and could yield some different solutions in comparison with known results in the literature. Due to the nature of random densities, results vary for different runs and sometimes best shapes are not symmetric, unlike the all-symmetry shapes of some problems in the literature. The proposal gives competitive results and shows the ability of BESO to find good solutions regardless of initial design meshes.

Key words: Bidirectional Evolutionary Structural Optimization, Topology Optimization

1. Introduction

Evolutionary structural optimization (ESO) [1] has been proposed for over two decades in an attempt to solve certain topology optimization problems. Originally, ESO uses von Mises stress or sensitivity number of strain energy [2] as a rejection criterion to remove the least effective parts during the evolution of the structure. Then it allows both the addition of necessary materials and the removal of unnecessary materials in a version called bidirectional evolutionary structural optimization (BESO) [3]. With efforts to upgrade its performance, there are some modifications are proposed, i.e., the use of mesh-independency filter and history information for stabilization of the evolutionary process in the ‘hard-kill’ BESO [4], combination with a material interpolation scheme [5] in the ‘soft-kill’ BESO [6-8], self-updating of evolutionary rate based on variation of the structure compliances [9], or an attempt of reaching a stationary state at each iteration in the evolutionary process [10]. More developments and application of ESO/BESO are reviewed in [11].

With the author’s knowledge, all the current versions of BESO start the evolutionary process from a solid pre-defined domain. In such approaches, the results are always identical if input parameters are the same. To check the ability of BESO with different initial design domains, we propose the use of random density between 0 (void) and 1 (solid) for the domain (0 < ρ < 1) instead of the fully solid density (ρ = 1). This study also utilizes the self-upgrading evolutionary ratio [9] and stationary steps in each iteration of volume [10] to solve benchmark problems. The structure of the article is as follows, A brief introduction of BESO is presented in section 2. The modification and test problems are given in sections 3 and 4. The last section is for some conclusions.

2. Bidirectional Evolutionary Structural Optimization for Compliance Minimization

This section describes briefly the ‘soft-kill’ BESO [6-8] applied to the problem of structural compliance minimization with given volume, as follows

\[
\text{Minimize } C = \frac{1}{2} u^T Ku
\]

Subject to \( V - V_0 = 0 \)

where \( C \) is the mean compliance of the structure; \( u \) and \( K \) are the nodal displacement vector and the overall stiffness matrix of the structure; \( V_0 \), \( V \) is the target volume and volume of the optimum structure, respectively. The volume of structure in finite element analysis is
\[ V = \sum_{i=1}^{n} w_i x_i \]  

(2)

where \( n \) is the total number of elements, \( w_i \) is the volume of the \( i^{th} \) element, \( x_i = 1 \) for solid elements, \( x_i = x_{\text{min}} \) for void elements. For comparison with results in the literature, values of \( x_{\text{min}} = 0, 0.01 \) are used.

Using material interpolation scheme with penalization \([5]\), Young’s modulus of the \( i^{th} \) element is defined as

\[ E(x_i) = E_e x_i^p \]  

(3)

where \( E_e \) is Young’s modulus of the solid element, \( p \) is the penalty exponent.

The evolutionary starts with an initial volume, normally the whole considered domain. The structure volume and topology will be changed at each iteration, as follows

\[ V_i = V_{i-1} (1 - ER) \quad k = 1, 2, 3, \ldots \]  

(4)

where \( ER \) is the evolutionary rate, \( k \) is the current iteration number.

At each iteration, the topology is updated using the value the sensitivity number \( a_i \) of elements. The sensitivity number \( a_i \) with regard to the change in the \( i^{th} \) element is calculated as

\[ a_i = \frac{1}{\rho} \frac{\partial C}{\partial x_i}, \quad \text{and} \quad \frac{\partial C}{\partial x_i} = -\frac{p x_i^{p-1} \mathbf{u}^T \mathbf{K} \mathbf{u}}{2} \]  

(5)

where \( \mathbf{u} \) and \( \mathbf{K} \) are the nodal displacement vector and the stiffness matrix of the \( i^{th} \) element.

For avoiding checkerboard and mesh-dependency, the filter of sensitivities is introduced. The modified elemental sensitivity is given as

\[ a_i = \frac{\sum_{j=1}^{M} w(r_{ij}) a_j}{\sum_{j=1}^{M} w(r_{ij})} \]  

(6)

where \( M \) is the total number of element centroid inside a circle of radius \( r_{\text{min}} \) centered at the centroid of the \( i^{th} \) element, and \( w(r_{ij}) \) is the linear weight factor defined by

\[ w(r_{ij}) = \begin{cases} r_{\text{min}} - r_{ij} & \text{for } r_{ij} < r_{\text{min}} \\ 0 & \text{for } r_{ij} \geq r_{\text{min}} \end{cases} \]  

(7)

where \( r_{ij} \) is the distance between centroids of the \( i^{th} \) element and \( j^{th} \) element, \( r_{\text{min}} \) is the filter radius.

For stability of the objective function convergence, sensitivity information of the previous iteration is used to average the current sensitivity as

\[ a_i = \frac{a_i + a_{i-1}}{2} \]  

(8)

where \( k \) is the current iteration number.

The design is updated by comparing the modified value \( a_i \) of the \( i^{th} \) element with a threshold of the sensitivity number \( a_o \) as follows

\[ x_i = \begin{cases} 1 & \text{if } a_i > a_o \\ x_{\text{min}} & \text{if } a_i \leq a_o \end{cases} \]  

(9)

where the threshold is set as the average sensitivity of the whole structure \( a_o = 0.5(a_{\text{avg}} + a_{\text{max}}) \).

The updated design is continually analyzed and the loop is stopped when \( V = V_h \) and the following convergence of the objective function is satisfied

\[ \left| \frac{\sum_{i=1}^{N} (C_{i+\tau} - C_{i-\tau+1})}{\sum_{i=1}^{N} C_{i+\tau}} \right| \leq \tau \]  

(10)

where \( k \) is the current iteration number, \( \tau \) is an allowable convergence error and \( N \) is set as 5 to ensure stable compliance in 10 consecutive iterations.

3. Proposed Modifications

3.1 Random Mesh

The modification of BESO is simply proposed as follows. For each run, BESO will start with a finite element mesh of
random density between void and solid (0 < \rho < 1) instead of a fully solid density, as shown in Figure 1. This proposal is expected to bring new solutions for different runs. It also aims to check the stability/convergence of method when it starts from meshes with different density. In this study, we also use other modifications of BESO from [9,10], as described in sections 3.2 and 3.3.

![Finite element mesh of a 2D problem](image)

Figure 1. Finite element mesh of a 2D problem, (a) Solid mesh, (b) Random density mesh

3.2 Inside Stationary Steps

For each iteration number \(k\), instead of only one topology update with the corresponding volume, the updated designs with the same volume are continually analyzed until a similar convergence as Eq. (10) is met. This modification showed the update designs are more stable in each iteration [10] and it also allows using larger values of the evolutionary rate \(ER\), e.g. \(ER=10\%\) in this study.

3.3 Self-adapting Evolutionary Rate

The adapting value of \(ER\) is based on the variation of the structure compliance in iterations. The updated value of \(ER\) for the next iteration is simply calculated as [9]

\[
\Delta ER = \frac{|C_k - C_{k-1}| - |C_{k-1} - C_{k-2}|}{C_k}
\]

\(ER_{k+1} = ER_k - \Delta ER\)

(11)

where \(k\) is the current iteration number (\(k \geq 3\)) and \(C_k\) is the structure compliance at the iteration \(k\).

With this adjustment, the evolutionary rate starts with an initial value \(ER\) and it will decrease if the change in structure compliance tends to increase and vice versa. It notes that the value of \(ER_{k+1}\) in Eq. (11) can be negative for a large increase of the compliance. In this case, the volume in Eq. (4) will be increased in the next iteration and the search can “go back” to explore a new path. This material addition is a different point in comparison with the “hard-kill” and “soft-kill” BESO where volume continually decreases until it reaches the target volume. It also notes that the update of \(ER\) is only used for the main loop with iteration \(k\), not for the proposed inside the loop of each volume as described in section 3.2.

4. Test Problems

Two common compliance minimization problems will be tested. Values of parameters in this test include the initial evolutionary rate \(ER_0 = 1\%\) to \(10\%\), the penalty exponent \(p = 3\), the filter radius \(r_{vol} = 1.5\) times element size and the allowable convergence error \(\varepsilon = 10^{-4}\). For each set of parameters, the problems will be run in 5 times with 5 random meshes.

4.1 Short Cantilever (5x8 cantilever)

A short cantilever is subjected to a concentrated load \(P = 100\) N at the middle of the free end as shown in Figure 2a. Material properties are Young’s modulus \(E = 100\) GPa and Poisson’s ratio \(\nu = 0.3\). The structure has meshed with 80 x 50 four-node plane stress elements. The mean compliance \(C\) of the structure, which has a target volume \(V_0 = 50\%\) of the whole domain volume, will be minimized.

From the results in Table 1, although fully solid meshes marginally outperforms random meshes in terms of compliance, not the iteration, there are some contrary cases. The best solution of a random mesh domain \(C = 1,8461\) Nmm is better than both the best results with a solid domain \(C = 1,8474\) Nmm in [10] and \(C = 1,8461\) Nmm in this study. Additionally, the use of random mesh can yield competitive results (bold numbers in Table 1) in comparison with the solid meshes. It is interesting that the optimal shape with a random mesh is not symmetric, unlike the
always-symmetry shapes with the initial solid mesh, as shown in Figure 2 b–c. It also notes that results are not considerably affected by different initial design meshes.

<table>
<thead>
<tr>
<th>ER (%)</th>
<th>Random mesh ((0 &lt; \rho &lt; 1))</th>
<th>Fully Solid Mesh ((\rho=1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>5 times of 1,8536 (129)</td>
<td>1,8507 (130)</td>
</tr>
<tr>
<td>1.5</td>
<td>1.8464 (81), 3 times of 1,8554 (83), 1,8523 (85)</td>
<td>1,8464 (85)</td>
</tr>
<tr>
<td>2.0</td>
<td>1.8461 (61), 1.8530 (59), 2 times of 1,8531 (59), 1,8523 (63)</td>
<td>1,8531 (63)</td>
</tr>
<tr>
<td>2.5</td>
<td>3 times of 1,8555 (47), 2 times of 1,8527 (47)</td>
<td>1,8467 (49)</td>
</tr>
<tr>
<td>3.0</td>
<td>4 times of 1.8527 (38), 1,8530 (59)</td>
<td>1,8482 (41)</td>
</tr>
<tr>
<td>3.5</td>
<td>1.8550 (31), 1.8553 (34), 1.8531 (31), 1.8525 (31), 1.8553 (34)</td>
<td>1,8521 (35)</td>
</tr>
<tr>
<td>4.0</td>
<td>1.8573 (27), 1.8541 (27), 1.8524 (28), 1.8565 (29), 1.8546 (31)</td>
<td>1,8544 (34)</td>
</tr>
<tr>
<td>5.0</td>
<td>1.8542 (29), 4 times of 1.8523 (63)</td>
<td>1,8523 (38)</td>
</tr>
<tr>
<td>10.0</td>
<td>1.8575 (16), 1.8660 (20), 1.8649 (19), 1.8754 (16), 1.8781 (16)</td>
<td>1,8675 (17)</td>
</tr>
</tbody>
</table>

Figure 2. Design domain of short cantilever (a). The optimal shapes of this study (b) and from [10]

4.2 Simple Beam (MBB Beam)

An MBB beam, shown in Figure 3a, is subjected to a concentrated load \(P = 10 \text{ N}\) at the bottom center of the span. Material properties are Young’s modulus \(E = 1 \text{ GPa}\) and Poisson’s ratio \(\nu = 0.3\). Due to symmetry, only the symmetric half has meshed with 120 x 40 four-node plane stress elements. The mean compliance \(C\) of the structure, which has a target volume \(V_s = 30\%\) of the whole domain volume, will be minimized.

<table>
<thead>
<tr>
<th>ER (%)</th>
<th>Random mesh ((0 &lt; \rho &lt; 1))</th>
<th>Fully Solid Mesh ((\rho=1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3.5838 (222), 3 times of 3.5711 (225), 3.6367 (226)</td>
<td>3.5838 (222)</td>
</tr>
<tr>
<td>1.5</td>
<td>2 times of 3.5751 (145), 3 times of 3.5908 (143)</td>
<td>3.5908 (143)</td>
</tr>
<tr>
<td>2.0</td>
<td>3.5743 (104), 3.5396 (109), 3.6104 (111), 3.5871 (107), 3.5898 (114)</td>
<td>3.5751 (106)</td>
</tr>
<tr>
<td>2.5</td>
<td>3.5954 (82), 3.5857 (86), 3.5449 (92), 3.5829 (87), 3.5924 (88)</td>
<td>3.5924 (88)</td>
</tr>
<tr>
<td>3.0</td>
<td>3.5376 (71), 3.5293 (68), 3.5408 (88), 3.5628 (72), 3.6058 (73)</td>
<td>3.6058 (73)</td>
</tr>
<tr>
<td>3.5</td>
<td>3.6018 (58), 3.5950 (59), 3.5865 (58), 3.6173 (64), 3.5867 (58)</td>
<td>3.5849 (60)</td>
</tr>
<tr>
<td>4.0</td>
<td>3.5189 (52), 3.6270 (51), 3.6035 (52), 3.6115 (68), 3.5695 (68)</td>
<td>3.5883 (55)</td>
</tr>
<tr>
<td>5.0</td>
<td>3.6225 (46), 3.6018 (45), 3.5847 (43), 3.5378 (45), 3.6356 (50)</td>
<td>3.5831 (50)</td>
</tr>
<tr>
<td>10.0</td>
<td>3.5982 (26), 3.6270 (45), 3.5821 (26), 3.6376 (26), 3.8766 (27)</td>
<td>3.8415 (26)</td>
</tr>
</tbody>
</table>
In this case, it is seen that the random meshes give better designs in most of cases. The best solution of a random mesh domain $C=3.5189 \text{ Nmm}$ is better than both the best results with a solid domain $C=3.5352 \text{ Nmm}$ in [10] and $C=3.5751 \text{ Nmm}$ in this study. Similarly, the use of random mesh can yield competitive results (bold numbers in Table 2) in comparison with the solid meshes. The optimal shape with a random mesh also differs from results with an initial solid mesh, as shown in Figure 3b-c. Again, the results are not considerably affected by different initial random meshes.

![Image](image_url)

Figure 3. Design domain of the MBB beam (a). The right half of the optimal shapes of this study (b) and from [10]

### 5. Conclusions

The proposal offers a new approach in exploring the problem of structural compliance minimization with BESO. The results of using random meshes as a starting design are competitive with those of using the solid mesh. One important thing is the approach shows the ability of BESO to find good solutions regardless of initial design meshes. Additionally, it can obtain better solutions in comparison with the current BESO versions using solid meshes although it needs some more trials. Besides, the optimal shapes are different with solutions in the literature. This approach can also be applied to other structural topology problems.

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A NON-LOCAL SIMP MODEL AND SENSITIVITY FILTERING

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Abstract

We present a rigorous mathematical model of topology optimization based on a non-local, integral, description of diffusion, which could for example describe steady state heat conduction. As is typically the case with other non-local models, we recover a local constitutive law in the limit of small non-local interaction horizons. In our case the limiting model is that of the local SIMP, thereby explaining the name non-local SIMP for the proposed model. Despite this close relationship to the standard (local) topology optimization with SIMP, our model enjoys several surprising properties not shared by its local counterpart. For example, the compliance minimization problem admits globally optimal designs for a range of penalty parameters of practical interest without needing any design regularization, even in the non-convex case. For a particular choice of SIMP exponent, the gradient of the reduced compliance functional bears close resemblance to the sensitivity filtering applied to the corresponding local model.

Key words: Non-Local Diffusion, Topology Optimization, SIMP, Existence of Solutions

1. Introduction

The utilization of homogenization-based topology optimization has nowadays become a standard part of engineering design process in mechanics and many other engineering applications, with compliance minimization based on SIMP material parametrization being arguably the most widely used model [1]. It is well understood, both theoretically and numerically, that the model does not attain its infimum without employing external regularization techniques, and an array of standard tools is readily available for this purpose [1]. In the earliest work on SIMP-based topology optimization a heuristic regularization, which mollifies the gradient of the objective function and not the set of admissible designs, also known as “sensitivity filtering”, has been employed with great success [2]. Sigmund and Maute [3] have recently attempted to interpret this regularization technique as the homogenization based topology optimization model, which is applied to the weakly non-local gradient plasticity model [3]. In the same spirit, one may wonder whether it is possible to come up with a plausible mechanical model, which incorporates a “self-regularization” mechanism with respect to homogenization-based topology optimization with SIMP. Following the ideas of [4-5] we present a mathematically rigorous model, which provides precisely such a construction.

Our model is similar to the one in [4] and also utilizes a derivative-free, integral description of diffusion phenomena. However, we use a different material parametrization model, which allows us to relate our non-local model to the standard local SIMP model, and therefore to answer the research question posed above.

Note that whereas diffusion is relevant for modelling many physical processes, such as steady state heat conduction, Darcy flow through porous medium, or small deflections of an elastic membrane, we will employ the vocabulary of heat conduction. Therefore, we will refer to the state of our system as “temperatures” and to the design parameters as “conductivities”.

2. Problem Formulation

2.1 Local Problem

Compliance minimization with SIMP applied to the standard, local heat conduction model can be described as follows. Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a bounded open domain with Lipschitz continuous boundary. We consider the material
distribution problem, where the design variable $\rho$ belongs to the class $\mathcal{A} = \{ \rho \in L^\infty(\Omega) \mid 0 < \rho \leq \rho_0 < \infty \} \subseteq \mathbb{R}$, for some constants $0 < \rho_0 \leq \rho_0 < \infty, 0 < \gamma < 1$. With each design $\rho$ we associate the conductivity $\kappa_c = \rho^r$, where $r \geq 1$ is a penalty parameter. In turn, the conductivity allows us to define the associated state of the system steady state temperature $u \in H^1(\Omega)$, which solves the variational formulation of the generalized Laplace problem:

$$
\int_\Omega \kappa_c(x) \nabla u(x) \cdot \nabla v(x) \, dx = \int_\Omega f(x) v(x) \, dx, \quad \forall v \in C^0(\Omega),
$$

where $f \in L^1(\Omega)$ is a given volumetric heat source. Finally, we are interested in minimizing $c_{\kappa_c}(\rho) = \int_\Omega f(x) u(x) \, dx$ over all admissible designs $\rho \in \mathcal{A}$.

It is well known that for $p = 1$ the problem is convex and optimal solutions exist [1, 6], whereas for $p > 1$ the problem is non-convex and the infimum is not attained [1]. In the latter context Sigmund [2] has proposed the following strategy. Let $c_{\kappa_c}(\rho, \xi) = -p \int_\Omega \rho^{p-1}(x) \xi(x) \left| \nabla u_{\kappa_c}(x) \right|^2 \, dx$ be the directional derivative of the “local” compliance in the direction $\xi \in L^\infty(\Omega)$, and let further $A: \Omega \times \Omega \rightarrow \mathbb{R}$ be an appropriate integral kernel. Where the optimality conditions for our topology optimization problem: $c_{\kappa_c}(\rho^*, \rho^* - \rho^*) \geq 0, \forall \rho \in \mathcal{A}$ may be unsolvable, there is strong numerical evidence that their perturbed version $c_{\kappa_c}(\rho^*, \rho^* - \rho) \geq 0, \forall \rho \in \mathcal{A}$, where $c_{\kappa_c}(\rho, \xi) = -p \int_\Omega A(x, y) \rho^{p-1}(x) \xi(x) \left| \nabla u_{\kappa_c}(x) \right|^2 \, dx \, dy$, admits solutions of practical relevance.

2. 2 Non-Local Problem

Let us first select the integral kernel $A(x, y)$ to be a radial function of $x - y$ supported on a small ball or radius $\delta > 0$. Parameter $\delta$ will play the role of the non-local interaction horizon between the particles in our system. We will require that $A(x, y) \sim \epsilon_{\text{norm}} |x - y|^{-d + 2 + 2\varepsilon}$ for small $|x - y|$, where $\epsilon_{\text{norm}} > 0$ is a normalization constant, and $\varepsilon \in (0, 1)$. The non-locality of interactions requires us to introduce a larger domain $\Omega = \{ x \in \mathbb{R} \mid \text{dist}(x, \Omega) < \delta \}$, which includes all points that may interact with points from $\Omega$. The variational formulation of the non-local diffusion problem can be, roughly speaking, obtained from that of the generalized Laplace problem by replacing the gradients with appropriately averaged finite difference approximations. Namely, the state $u \in L^1(\Omega)$, $u = 0$, in $\Omega \setminus \Omega$, of our system will satisfy the equation

$$
\int_{\Omega \setminus \Omega} \kappa(x, y) A(x, y) \frac{u(x) - u(y)}{|x - y|} \frac{u(x) - u(y)}{|x - y|} \, dx \, dy = \int_\Omega f(x) v(x) \, dx, \quad \forall v \in C^0(\Omega).
$$

Note that the equation above includes the material parameter $\kappa(x, y)$ describing the non-local conductivity between material points $x$ and $y$. In this work we assume that this conductivity is simply the geometric mean of the local conductivities satisfying the SIMP model: $\kappa(x, y) = \kappa_{\text{L}}^\varepsilon(x) \kappa_{\text{L}}^\varepsilon(y) = \rho^{2\varepsilon}(x) \rho^{2\varepsilon}(y)$. Similarly to the local case, we will be interested in minimizing the compliance $c_{\kappa_c}(\rho)$ defined as the average temperature weighed with volumetric heat source.

The only difference in the definition of our problem compared to the one studied in [5] is that we utilize a geometric mean when defining the non-local conductivity, whereas in [5] the arithmetic mean is used. However, this small difference leads to some very favourable properties in the topology optimization context.

3. Main Theoretical Results

3.1 Existence of Optimal Solutions

Similarly to the local case, the compliance minimization model for the non-local diffusion equation is convex for $p = 1$ and non-convex for $p > 1$. However, it turns out that it attains its infimum even in some non-convex cases. Namely, we have the following result.

**Theorem 1.** For each $p \in [1, 2]$, the non-local compliance minimization problem admits a globally optimal solution.

3.2 Relation to the Local Problem

For a very large class of material distributions, including all those which can be computed numerically, we have the following result.

**Theorem 2.** Let $A_{\text{single}} = \{ \rho \in A \mid \rho(x) = \sum_{i=1}^N \rho \chi_a(x) \}$, where $\chi_a$ are characteristic functions of some open
subsets of $\Omega$, and $\rho$ are some constants. If $\rho$ can be approximated pointwise with designs from $A_{\text{simple}}$, then

1) $\lim_{\mu \to 0} u_\rho(\rho) = u_0(\rho)$, and 2) $\lim_{\mu \to 0} c_\rho(\rho) = c_\text{loc}(\rho)$. That is, non-local temperatures converge to the local temperature as defined by the generalized Laplace equation with SIMP for the given design, and non-local compliance values converge to the local compliance value.

Note that the class $A_{\text{simple}}$ defined in Theorem 2 is very large indeed; for example it includes all piece-wise constant finite element functions on all meshes, and all their pointwise limits. For practical purposes it includes all “computable” designs.

**Theorem 3.** Let $\rho_k \in A$ be a sequence of designs, such that $\rho_k \to \rho'$, that is the local conductivities defined by these designs converge, weakly-$*$ in $L^\infty(\Omega)$. Then $c_\text{loc}(\rho) \leq \lim_{\mu \to 0} c_\rho(\rho)$.

Together, Theorems 2 and 3 are reminiscent of $\Gamma$-convergence results, which would guarantee that global minima of approximating (in our case, non-local) problems converge to the global minimum to the limiting (local) problem. Of course, $\Gamma$-convergence for $p > 1$ is impossible, as the limiting problem does not admit globally optimal solutions. Nevertheless, the relation between the two problems is undeniable and is clearly seen in our numerical experiments.

### 3.3 Relation to Sensitivity Filtering

Differenitizing the non-local compliance we obtain the following expression for the directional derivative: $c_\rho(\rho; \xi) = -\frac{\partial}{\partial \rho} \int_{\Omega} [\rho^{p-1}(x)\rho^{p-1}(y)\xi(x) + \rho^{p-1}(x)\rho^{p-1}(y)\xi(y)] A(x,y) \left| \frac{u(x) - u(y)}{|x-y|} \right|^2 dx dy$. In particular case $p=2$ we have $c_\rho(\rho; \xi) = -2 \int_{\Omega} A(x,y) \rho(x)\xi(y) \left| \frac{u(x) - u(y)}{|x-y|} \right|^2 dx dy$, which bears striking resemblance with the expression for the “filtered sensitivities” for $p=2$. Indeed, in the same way as the non-local state problem is obtained by replacing the gradients with averaged finite differences, so is the directional derivative for the non-local compliance is obtained from the sensitivity filtered derivative of the local compliance. Unfortunately, in the limit of small non-local interaction horizons the filtering disappears, and only the non-regularized expression remains.

### 4. Numerical Experiments

#### 4.1 Solution of the State Equations

The symmetric coercive variational formulation of the non-local state problem naturally fits into the Galerkin-Ritz framework. Therefore, we employ standard piecewise-linear polynomials on a simplicial mesh for discretizing the non-local governing equations. Unfortunately, owing to the non-locality of the bilinear form, the work and memory requirements on a quasi-uniform grid grow as $O(\delta h^{-2})$, because one has to integrate over all pairs of simplices, which are “close enough” to each other as defined by the non-local interaction horizon. Furthermore, owing to the singularity of the interaction kernel $A(x,y)$, one has to utilize special quadratures, which avoid cancellation errors for small $|x-y|$. In our implementation we utilize the quadratures described in [7]. Since the convergence theory of [7] does not precisely apply to our situation, we verify numerically their exponential convergence for evaluating elemental stiffness matrices, as well as estimate how many quadrature points we need to achieve nearly full IEEE double precision accuracy and therefore to avoid variational crimes. It turns out the number of quadrature points is quite staggering: in the most singular situation (corresponding to integration over the same simplex, so $|x-y|$ is always small) we need about $7, 6 \cdot 10^7$ points per pair of simplices in dimension $n=2$, which then corresponds to a four-dimensional integral. Owing to such a high cost of integration we focus the development on structured grids only, where we can pre-assemble elemental matrices for a small range of the mesh (corresponding to the support of the kernel) and utilize them throughout the rest of the grid. Still, the high memory requirements preclude us from utilizing sparse direct solvers for all but the tiniest of problems, but AMG-accelerated CG works reasonably well. We have tested our implementation using the method of manufactured solutions.

#### 4.2 Numerical Illustration of Theorem 2

For a given smooth design (we use a Gaussian distribution), $p=2$, and a prescribed smooth local temperature we compute the corresponding volumetric source $f$. This source is then used to solve a range of discretizations of non-local problems with varying non-local horizon $\delta$ and mesh element size $h$. We then compare the obtained numerical
approximation of the non-local steady state temperature with a known local analytical solution. The results are summarized in Figure 1. Our main observation is that we observe convergence predicted by Theorem 2, but unsurprisingly we need finer meshes to properly resolve non-local problems with smaller values of δ.

![Figure 1. Numerical illustration of theorem 2](image)

**4.3 Solving the Optimization Problem**

We utilize the standard optimality criterion method for solving the non-local compliance minimization problem, although other alternatives are clearly possible. We use $Ω=(0,1)^2$, $f=1$, $s=1/3$, $p \in \{1,2\}$, $δ \in \{0.05,0.1,0.2\}$. In the convex case $p=1$ we also perform a comparison with the local problem, which we mark as $δ=0$. The results for the convex case $p=1$ are summarized in Table 1 and Figure 2, and for the non-convex case $p=2$ in Table 2 and Figure 3.

<table>
<thead>
<tr>
<th>δ</th>
<th>h</th>
<th>Compliance</th>
<th># OC iterations</th>
</tr>
</thead>
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<tr>
<td>0.2</td>
<td>8.84E-03</td>
<td>9.24E-02</td>
<td>27</td>
</tr>
<tr>
<td>0.2</td>
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<tr>
<td>0.1</td>
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<td>7.99E-02</td>
<td>32</td>
</tr>
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<td>0.1</td>
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<tr>
<td>0</td>
<td>3.13E-03</td>
<td>6.95E-02</td>
<td>38</td>
</tr>
</tbody>
</table>

![Figure 2. Optimal conductivity distributions for $p=1$. Left: non-local problem, $δ=0$. Right: local problem](image)
Table 2. Summary of the Results for $p=2$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$h$</th>
<th>Compliance</th>
<th># OC iterations</th>
</tr>
</thead>
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<td>0.05</td>
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</tr>
</tbody>
</table>

Figure 3. Found stationary (non-convex problem) conductivity distributions for $p=2$. Left: non-local problem, $\delta=0.2$; right: non-local problem, $\delta=0.05$.

5. Conclusions

We present a topology optimization framework for a model governed by a non-local diffusion equation. In contrast with the behavior of the local problem, the infimum for the non-local problem is attained even in the non-convex case corresponding to SIMP exponents $p \in (1, 2]$. The gradient of the compliance resembles “sensitivity filtering” for the choice of $p=2$, thus providing a rational explanation for the empirical regularizing effect of this technique on the local compliance minimization problem with SIMP.

Acknowledgments

AE’s research is financially supported by the Villum Fonden through the Villum Investigator Project InnoTop. AE is grateful to Alexey Chernov for a helpful discussion of singular integration on products of simplices. The work of JCB is funded by FEDER EU and Ministerio de Economía y Competitividad (Spain) through grant MTM2017-83740-P.

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DENSITY-BASED TOPOLOGY OPTIMIZATION USING THE ERSATZ MATERIAL MODEL

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Abstract
This paper presents a density-based floating projection topology optimization (FPTO) method using the ersatz material model without material penalization. The formation of a solid/void topology attributes to the imposed constraint of the floating project rather than the material penalization scheme in the well-established density-based topology optimization methods, such as the SIMP and BESO. The effectiveness of the proposed method has been demonstrated on a series of numerical examples for compliance minimization. The proposed FPTO method retains the advantages of the density-based topology optimization method and avoids the potential difficulties in the selection of the penalty exponent.

Key words: Topology Optimization, Floating Projection, Ersatz Material Model

1. Introduction
The Structural topology optimization has received considerable attention in the last several decades and many topology optimization methods have been developed since the seminal paper by Bendsøe and Kikuchi [1] in 1988. Generally, the well-established gradient-based topology optimization methods can be classified into two categories according to the definition of design variables, density and boundary methods.

The popular density-based topology optimization methods include the solid isotropic material with penalization (SIMP) method [2-5] and the modern gradient-based bi-directional evolutionary structural optimization (BESO) [6-9]. The core of the density-based topology optimization methods lies in the artificial power-law relation between the density of an element and its material property. When the penalty exponent is large enough, a clear topological design can be obtained. The adverse effect is that the solution may depend on the selection of the penalty exponent. The representative boundary-based topology optimization method is the level-set method, which enables to capture the change of topology by merging holes [10-11]. The ersatz material model without material penalization has been widely adopted for the level-set method. So far, there is no density-based topology optimization using the ersatz material model, which can potentially retain the advantages of the density-based topology optimization method and eliminate the adverse impact of the material penalization scheme.

Unfortunately, the optimization problem using the ersatz material model corresponds to the so-called “variable-thickness-sheet” problem and the resulting design contains a large volume of “grey area”. In order to generate a clear 0/1 topology, this paper introduces the constraint of the floating projection to push the design variables to their upper and lower bounds. The effectiveness of the proposed method will be demonstrated through a series of numerical examples for compliance minimization.

2. Topology Optimization Framework
Topology optimization is usually based on finite element analysis (FEA), where the design domain is discretized into a number of finite elements \((i=1,2,\ldots,M)\). Taking a typical compliance minimization of a structure as an example, the topology optimization problem can be expressed by the following equation within the framework of FEA.
\[
\begin{align*}
\min_{C} & = f^* u \\
Ku & = f \\
V_j & = \sum_{\rho} \rho V_i \\
\rho & \in \{0, 1\}
\end{align*}
\] (1)

Where \(C\) is the compliance of the structure, \(f\), \(K\) and \(u\) are the force vector, stiffness matrix and displacement vector, respectively, \(E\) is the Young’s modulus of element \(i\), \(V_i\) and \(V_j\) are the volume fraction of the design and its constraint value, respectively, \(V_i\) is the volume of element \(i\), \(\rho\) is the density (or thickness in 2D case) of element \(i\), and \(\rho = 0\) or \(1\) denotes that element \(i\) is a void or solid element, respectively. Because the design variable, \(\rho = (\rho_1, \rho_2, \ldots, \rho_n)\), only takes discrete values 1 or 0, the above topology optimization is difficult to be solved mathematically. To circumvent this problem, design variables can be relaxed to be continuous. For the density-based topology optimization methods, the fixed mesh is normally used to avoid the re-meshing in each iteration. Hence, the topology optimization problem defined in eq. (1) will result in a design with zig-zag boundaries, which is impractical for applications. Therefore, the proposed topology optimization method seeks a nearly 0/1 solution which allows the existence of intermediate design variables at boundaries in order to generate a smooth design.

Consider the upper and lower bounds of design variables, \(0 \leq \rho_{\text{max}} \leq \rho \leq 1\) is firstly assumed, where \(\rho_{\text{max}}\) assigns with a small value, e.g., \(10^{-6}\) to avoid the possible singularity of the problem. For the design variables between \(\rho_{\text{min}}\) and 1, the ersatz material model [10-11] without material penalization is adopted as

\[E(\rho) = \rho E^i\] (2)

where \(E^i\) denotes the Young’s modulus of the solid. For finite element analysis, the elemental stiffness matrix can be expressed by

\[K(\rho) = \rho K^i\] (3)

where \(K^i\) denotes the stiffness matrix of element \(i\) when it is full of the solid.

In order to form a clear topology, two extra constraints, the density filter and floating projection, will also be imposed in this paper. The objective function of the original problem can be modified by introducing Lagrange multipliers

\[
\min_{\rho, f} = \frac{C}{\lambda} + \sum_{i} \sum_{j} \lambda_i (g_i - g_i^j)
\] (4)

where \(g_i\) and \(g_i^j\) denote the constraint and its constraint value on the design variables including the upper and lower bounds, the density filter and floating projection, \(\lambda > 0\) and \(\lambda_i > 0\) are the Lagrange multipliers of the corresponding constraints. Since the upper and lower bounds, density filter and floating projection will be strictly enforced in the update scheme and the last term in eq. (4) can be ignored.

The above minimization problem can be equivalently solved by using a bounded Lagrange multiplier \(\lambda\) for the volume constraint

\[
\min_{\rho, f} = (1-\lambda) f + \lambda (V_j - V_j^i)
\] (5)

where \(0 \leq \lambda < 1\); \(\lambda\) will be determined according to the volume constraint which will be explained later. The ideal condition for an optimal design is that all design variables are located at the stationary point of the modified objective function as

\[
(1-\lambda) \frac{\partial C}{\partial \rho} + \lambda \frac{\partial V_j}{\partial \rho} = 0
\] (6)

where \(\frac{\partial C}{\partial \rho} = -u^T K u\), and \(\frac{\partial V_j}{\partial \rho} = \frac{V_i}{\sum V_i}\).

To damping the update of design variables, the elemental sensitivities can be averaged with their values in the previous steps as

\[
\frac{\partial C_k}{\partial \rho} = \frac{\partial (C_{k-1})}{\partial \rho} + \frac{\partial C_k}{\partial \rho}
\] (7)

where the superscript \(k\) denotes the current iteration number.

The optimality condition in eq. (6) can be rewritten as

\[
\lambda = (1-\lambda) \frac{\partial C}{\partial V_j} + \frac{\partial C_k}{\partial V_j} = 0
\] (8)

To seek a minimum of the modified objective function in eq. (5), the design variables can be updated according to the steepest descent principle by enforcing their lower and upper bounds
\[ \rho = \begin{cases} 
\rho_{\text{min}} & \text{if } \rho^{(k-1)} (1-A) \leq \rho_{\text{min}} \\
\rho^{(k-1)} (1-A) & \text{otherwise} \\
1 & \text{if } \rho^{(k-1)} (1-A) \geq 1 
\end{cases} \]  
(9)

where the negative sign before \( A \) denotes the search direction.

Topology optimization of continuum structures often encounters numerical instabilities, e.g., the checkerboard pattern and mesh-dependence. The heuristic filter scheme had demonstrated its ability to overcome those problems with low computational cost. The density filter essentially imposes a heuristic minimum size constraint of the original problem, and modifies the design variables as

\[ \hat{\rho} = \frac{\sum w(r_{ij}) \rho_i}{\sum w(r_{ij})} \]  
(10)

where \( r_{ij} \) denotes the distance between the centers of the \( i \)th and \( j \)th elements, \( w(r_{ij}) \) is the weight factor defined by

\[ w(r_{ij}) = \begin{cases} 
\frac{r_{\text{min}} - r_{ij}}{r_{\text{in}}} & \text{if } r_{ij} \leq r_{\text{in}} \\
0 & \text{if } r_{ij} > r_{\text{in}} 
\end{cases} \]  
(11)

where \( r_{\text{in}} \) is the filter radius specified by the user.

In order to obtain an accurate topological design, this paper proposes the constraint of the floating projection, which further modifies the design variables as,

\[ \rho^k = \frac{\tanh(\beta \cdot th) + \tanh(\beta \cdot (\rho^k - th))}{\tanh(\beta \cdot th) + \tanh(\beta \cdot (1-th))} \]  
(12)

where \( \beta > 0 \) controls the steepness of the Heaviside function. The non-prior threshold \( th \) can be easily determined by ensuring \( \sum \rho^k = \sum \rho^* \). Different from the fixed Heaviside projection \( th = 0, 5 \) [10], the threshold value of the floating projection is determined through the competition of all design variables and will be updated step by step, \( \rho^k \) is the updated design variables, which will be used for the next finite element analysis. For some cases, one may instead use a moving limit \( \delta \), so that \( \rho_{\text{max}} \leq \rho^k \leq \rho_{\text{max}}(1-\delta) \leq \rho^k \leq \rho_{\text{max}}(1+\delta) \leq 1 \). It should be also noted that the original projection technique itself [12-14] has no ability to generate a clear 0/1 design.

Undoubtedly, a larger \( \beta \) makes the design variables even closer to solid/void, but may lead a local optimum due to the non-convex nature of topology optimization. Using the continuation concept, \( \beta \) starts from an initial value \( \beta = 10^{-6} \) and then increases with \( \Delta \beta \) (e.g., 0.5 in this paper) once a convergent solution is achieved. Thus, the floating projection gradually tunes intermediate design variables toward their upper and lower bounds as \( \beta \) increases, \( \beta \) will be stopped to increase once the solution is close to a 0/1 design.
One unresolved problem is on the calculation of the Lagrange multiplier $0 \leq \lambda < 1$, according to the volume constraint. This can be accomplished by the bi-section method or the Newton method (also known as the Newton-Raphson method). The bi-section method can conveniently handle with a sole constraint, but hardly for multiple constraints. Here, we introduce the Newton method for the calculation of the Lagrange multiplier according to the volume constraint. Generally, the volume constraint in the kth iteration is

$$ g = V_f - V_j = \sum \hat{\rho}_i \frac{\partial V}{\partial \rho_i} - V_j \leq 0 $$

(13)

Due to the variation of the design variables, the constraint function can be estimated by

$$ g^{(k-1)} + \sum \frac{\partial g}{\partial \rho} \frac{\partial \rho}{\partial \hat{\rho}} \frac{\partial \hat{\rho}}{\partial V} (\hat{\rho} - \rho^{(k-1)}) = \sum \frac{\partial g}{\partial \rho} \frac{\partial \rho}{\partial \hat{\rho}} \frac{\partial \hat{\rho}}{\partial V} \rho^{(k-1)} (1 - \lambda) \left( 1 - \frac{\partial C/\partial \rho}{\partial V_f/\partial \rho} \right) $$

(14)

$g^{(k-1)}$ is the unbalanced constraint value resulting from the last iteration. Thus, the gradient of $\lambda$ can be expressed by

$$ \frac{\partial g}{\partial \lambda} = - \sum \frac{\partial g}{\partial \rho} \frac{\partial \rho}{\partial \hat{\rho}} \frac{\partial \hat{\rho}}{\partial V} \rho^{(k-1)} (1 - \lambda) \left( 1 - \frac{\partial C/\partial \rho}{\partial V_f/\partial \rho} \right) $$

(15)

Then, the Newton-Raphson procedure will be used to update $\lambda$ iteratively as

$$ \lambda^i = \lambda^{i-1} - \frac{g}{\partial g/\partial \lambda} $$

(16)

where $i$ is the iteration number of the inner loop. The Newton-Raphson procedure starts from an initial guess value such as $\lambda^0 = 0$, and the loop will be stopped until the variation of $\lambda$ is small enough, e. g., $0.0^\circ$. Due to the introduced bounds of $\lambda$, the computational cost becomes negligible.

An ideal solid/void design can be assumed to be the one that $\rho = 1$ for $\rho \geq th$ and $\rho = \rho_{tim}$ for $\rho < th$. To represent the optimized topology with smooth boundary, the densities of elements, $\rho$, can be linearly interpolated into the whole design domain, $\rho(x,y)$. Then, we construct a level-set function $\phi(x,y) = \rho(x,y) - th$ and an optimized topology can be represented by the zero level-set as

$$ \begin{align*}
\phi(x,y) &> 0 & \text{for solid region} \\
\phi(x,y) &= 0 & \text{for boundary} \\
\phi(x,y) &< 0 & \text{for void region}
\end{align*} $$

(17)

The whole optimization procedure is summarized as the flowchart in Figure 1.

### 3. Numerical Examples

In this section, several topology optimization problems for compliance minimization are presented. In all examples, the solid material is assumed with Young’s modulus of 1MPa and Poisson’s ratio of 0.3. The optimization parameters are filter radius $r_{min} = 2$ and move limit $\delta = 0.2$.

#### 3.1 A Short Cantilever

This example considers the compliance minimization of a short cantilever shown in Figure 2. The design domain is $60 \times 40$, which is meshed with $60 \times 40$ four-node plane-stress elements. The available material for constructing the final design will cover 50% of the whole design domain.

The optimization starts from a guess design composed of elements with $\rho = 0.5$. The history of compliance during optimization is shown in Figure 3. The compliance initially decreases to a minimum value but the topology contains a large volume of intermediate elements. When $\beta$ increases, the topology evolves toward a solid/void design and meanwhile the compliance gradually increases. Finally, the compliance is convergent at 31.59. The final optimized design is shown in Figure 4(a) represented by elements and Figure 4(b) by the level-set function.

To verify the proposed FPTO method, the above problem using the same mesh is solved using the well-established continuation SIMP method. In order to achieve a nearly solid/void design, the penalty exponent $p (\Delta p = 0.5)$ gradually increases from 1. Meanwhile, the filter radius $r_{min} (\Delta r_{min} = 0, 2)$ gradually decreases from 2 to 1.1. The histories of penalized and unpenalized compliances for the SIMP method are also given in Figure 3 for comparison. For the first eight iterations, the resulting compliances from the FPTO and SIMP methods are almost the same. When
p suddenly increases, the penalized compliance jumps to a high value and then decreases. These jumps attribute to the large increment of the penalty exponent, which greatly overestimates the compliance of the design. However, the history of unpenalized compliance is very similar to that of the proposed FPTO method. That is, the compliance increases as the solution tends to a solid/void design. The unpenalized compliance of the SIMP design is 31.68, which is very close to that of the FPTO design. Meanwhile, the final SIMP topology shown in Figure 4(c) is almost identical to the one obtained by the FPTO method. This example also demonstrates that the selection of the penalty exponent greatly affects the calculation of the displacement at the force point.

![Figure 4. Comparison of optimized topologies for a short cantilever: (a) The FPTO method represented by elements; (b) The FPTO method represented by the level-set function; (c) The continuation SIMP method by elements](image)

### 3.2 Michell Type Structures

Michell type structures have been widely used for verifying the effectiveness of an optimization algorithm. Figure 5(a) shows the symmetric right half of an MBB beam with 150 in length and 50 in height. The design domain has meshed with 150×50 four-node plane-stress elements. The volume is restricted to be 60% of the whole design domain. The final design is shown in Figure 5(b). The compliance of the optimized design is stably convergent at 158.60 as shown in Figure 5(c).

![Figure 5. (a) Design domain; (b) Optimized design; (c) Convergence history of compliance](image)

Next, we examine a long cantilever as shown in Figure 6(a). The design domain has also meshed with 150×50 four-node plane-stress elements and the volume fraction is restricted to be 60%. The optimized topology is shown in Figure 6(b). Figure 6(c) shows the convergence history of compliance through the whole optimization. The final compliance of the optimized design is 145.48.

![Figure 6. (a) Design domain; (b) Optimized design; (c) Convergence history of compliance](image)

The optimized topologies in all those cases bear a strong resemblance to the original Michell truss as well as the
optimized designs in literature [4, 12, 14], which well verify the effectiveness of the proposed method.

3. 3D Structures

The proposed topology optimization method can be equally extended to 3D cases. Figure 7 (a) shows the design domain of a 3D cantilever, which is discretized by $60 \times 40 \times 10$ brick elements. The final volume is restricted to be 20% of the whole design domain. Figure 7 (b) shows the optimized design expressed by the level-set function. Figure 7 (c) shows the history of compliance during the optimization. Similar to 2D, the compliance initially decreases to its minimum, then slightly increases as the solution evolves toward a solid/void design. At the final stage, the compliance is stably convergent at 4.09.

(a)  (b)  (c)

![Figure 7](image)

Figure 7. (a) Design domain; (b) Optimized design; (c) Convergence history of compliance

Figure 8(a) shows the design domain of a 3D bridge undergoing a uniform pressure. The design domain has meshed with cubic elements of size $1 \times 1 \times 1$. Elements for the top three layers acting as the deck of the bridge are assumed to be solid and non-designable. The middle slot is designated for the passage of ships. The volume fraction is restricted to 20%. The final design is shown in Figure 8 (b).

![Figure 8](image)

Figure 8. The design domain of a 3D bridge

4. Conclusions

This paper proposes a new density-based topology optimization method using the ersatz material model where the relaxed design variables are constrained by the density filter and the floating projection. Most importantly, the constraint of the proposed floating projection pushes the design variables to their upper and lower bounds by gradually increasing the steepness of the projection, $\beta$. The effectiveness of the proposed FPTO method has demonstrated on a series of 2D and 3D numerical examples. The proposed FPTO method retains the advantages of the density-based topology optimization method and eliminates the adverse impacts of the material penalization scheme. The further research will apply this method to complex multi-physics topology optimization problems.

Acknowledgements

The author wishes to acknowledge the financial support from the Australian Research Council (FT130101094).

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PARAMETERIZED LEVEL-SET BASED TOPOLOGY OPTIMIZATION FOR CONTINUOUS STRUCTURES WITH SELF-WEIGHT

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Abstract

For a large-scale structure, self-weight is one of the main loads that must be considered in the design. However, the particular difficulty of the parasitic effect for low densities arises in the considered topology problems when using the popular power model (SIMP). Therefore, some specific treatments must be applied to overcome this difficulty but which may generate an ill effect on the optimal solutions. This paper studies topology optimization for continuous structures with self-weight based on the parameterized level-set method using Radial Basis Functions (RBFs). Since there is no need to involve penalization, this problem can be naturally avoided in the parameterized level-set based model. A numerical example of a rectangle plate design is studied and the solutions from several different methods are compared and discussed. The comparison shows the result of the proposed method has a different configuration but the minimum objective. In addition, by applying a stress criterion, an interesting observation is found that the solution is quite similar with the result obtained with the SIMP method.

Key words: Topology Optimization, Self-Weight, The Parameterized Level-Set Method

1. Introduction

The structural topology optimization design can be realized by solving an optimal distribution problem of materials in the design space [1]. The structural optimization design can not only save the cost of resources, but also optimize the quality function of the structure, improve the utilization of space, and maximize the effectiveness of resources. The popular methods include the method of Solid Isotropic Material/Microstructure with Penalization (SIMP) [2], Evolutionary Structural Optimization (ESO) method [3], the level-set method [4] and so on. With the developments in the topology optimization theory, many engineering problems have been successfully addressed by this conceptual design tool.

In the literature related to the topology optimization of continuum structures, the self-weight load of the structure is often neglected. However, as for the large-scale structures, the structure weight is one of the main loads that should be considered in the design. In topology optimization of those structures, the presence or absence of self-weight is directly related to the presence or absence of the material at each point in the design. This is a typical design-dependent problem, which means the consideration of the self-weight in the structural topology optimization is not just a simple addition of self-weight in finite element analysis. It is different with the situation when only a fixed external load is considered, in which the increase in the amount of material only serves to increase the structural stiffness, so that the structural strain energy is always reduced. When considering the self-weight, the increase in material consumption will also lead to the augmentation of loads, and the increase or decrease in structural strain energy will vary from case to case. Therefore, this feature needs to be considered and processed in the structural topology optimization method.

In 1999, Torteltaub and Washabaugh [5] proposed that the volume constraint should be loosely constrained after considering the design-dependent physical strength in the problem of structural strain energy minimization. In 2001, Bruyneel and Duysinx [6] proposed that it is simple to introduce design-dependent physical strength from the perspective of sensitivity analysis considering the self-weight. However, the previous method adopted to solve the
problem gives rise to the optimization convergence problem and parasitic effect in low-density regions. This problem refers to the fact that the low-density material is not stiff enough to support its own weight, which may lead to excessive local deformation so that the low-density region with optimized results needs to compensate for the distribution of gray intermediate density material to support the low density-region. In 2002, Bruyneel and Duysinx [7] proposed that GBMMA (Gradient-based Method of Moving Asymptotes) can avoid the turbulence problem caused by the non-monotonicity of the objective function in the optimization process. Yang et al. [8] used evolutionary structural optimization (ESO) and bi-directional evolutionary structural optimization (BESO) respectively to study the self-weight and surface load of load-transfer structures. Lopes et al. [9] studied the topological optimization design of three-dimensional structures subjected to self-weight and inertia load. Nevertheless, to the current knowledge of the authors, there has not been much research on this problem in the level-set based topology optimization framework.

In this paper, the authors study the topology optimization problem for continuous structures with self-weight based on the parameterized level-set method using Radial Basis Functions (RBFs). Firstly, the particular difficulties arising in the considered topology optimization problem are pointed out, non-monotonous behavior of the compliance, the possible unconstrained character of the optimum, and parasitic effect for low densities. In addition, because the optimal results of SIMP and ESO are significantly different in the previous literature, the level-set method is used to study the problem and explore the reason. Considering the operational simplicity and smooth and clear boundary representation during the optimization process, the parameterized level-set method is chosen. Finally, numerical examples are provided to illustrate the effectiveness of solving self-weight problems.

2. Parameterized Level-Set Based Topology Optimization for Continuous Structures with Self-Weight

In the level-set based topology optimization framework, a two-dimensional structure can be fitted to a smooth one dimensional higher level-set surface, and the expression of the structure’s boundary can be performed by the zero contour lines of the level-set function \( \phi(x,t) \):

\[
\begin{align*}
\phi(x,t) > 0 \quad & \forall x \in \Omega \setminus \partial \Omega \\
\phi(x,t) = 0 \quad & \forall x \in \partial \Omega \\
\phi(x,t) < 0 \quad & \forall x \in D \setminus \Omega
\end{align*}
\]  

(1)

where \( x \in \mathbb{R}^2 \) or \( \mathbb{R}^3 \) denotes a point in the design domain \( D \subset \mathbb{R}^2 \) or \( \mathbb{R}^3 \), and \( \Omega \) and \( \partial \Omega \) are the solid part and the boundary of the object respectively. \( t \) is the pseudo time to represent the updating iteration steps during the evolution of the level-set function. In the conventional manner, the Hamilton-Jacobi equation, given in Eq. (2), is used to describe the evolution of the level-set function.

\[
\frac{\partial \phi}{\partial t} - V_n |\nabla \phi| = 0
\]  

(2)

Where \( V_n \) is the external normal velocity along the boundary, which is defined by Eq. (1).

In some variational level-set model, such as the parameterized level-set method [10], the updating equation of the level-set function is replaced by an ordinary differential equation as shown in Eq. (3),

\[
\frac{\partial \phi}{\partial t} - V_n \delta (\phi) = 0
\]  

(3)

Where \( \delta (\phi) \) is a Dirac delta function of \( \phi \) [10].

The parameterized level-set based topology optimization was firstly proposed by Wang et al. in 2007 [10-11]. Compared with the conventional level-set method, the key improvement of this method is to represent the level-set function \( \phi(x,t) \), which generally is a scalar data field described by scattered points in the traditional level-set method, by a linear combination of a set of Radial Basis Functions (RBFs) \( m_i(x) \). There are some commonly used RBFs such as Compactly Supported Radial Basis Functions (CSRBFs), Multiquadric (MQ) splines and Inverse Multiquadric (IMQ) splines. In this study, the MQ splines, as given in Eq. (4), are used to construct the level-set function.

\[
m_i(x) = \sqrt{\left( x - x_i \right)^2 + c} \quad , \quad x_i \in D
\]  

(4)

Where \( x_i \) is the coordinates of the \( i \)th knot which indicates the location of the basis function, and \( c \) is the corresponding shape parameter which is commonly assumed to be a constant with a small value for all knots in most applications. The parameterized level-set function \( \phi(x,t) \) is represented by \( n \) MQ splines centered at \( n \) fixed knots and
can be written as:

\[ \varphi(x,t) = \sum_{i=1}^{n} a_i(t) m_i(x) + p(x,t) \]  

(5)

where \( a_i(t) \) is the coefficient of the radial basis function located on knot \( i \), \( p(x,t) \) is a linear polynomial to keep the positive definiteness of the solution. After combining with topology optimization based on the level-set method, RBFs can be used to reconstruct and optimize structural topology with any possible manifold. Beyond that, the Hamilton-Jacobi partial differential equation (PDE) is transformed into a relatively simple ordinary differential equation (ODE). It can not only retain the advantages of implicit boundary description with the level-set method, but also avoid some numerical difficulties caused by solving the PDE in discretized meshes.

By substituting Eq. (5) into the ordinary differential equation Eq. (3), the vector form of the coefficients \( a \) can be updated by the following equation [10]:

\[ a(t_{k+1}) = a(t_k) + \Delta t \delta(\varphi) M^{-1} V_e \]  

(6)

where \( k \) denotes the iterative step number, \( \Delta t \) the time step size, \( M \) an invertible matrix composed by the basis functions \( m_i \) and \( V_e \) the velocity vector. The detailed derivation is given in [10].

In Eq. (6), \( V_e \) is determined according to the sensitivity analysis of the mathematical model of a specified topology optimization problem. A compliance minimization problem in the level-set based framework can be defined as:

\[ \min_{\varphi} J(u,\varphi) = \int_D \varepsilon^T(u) : C : \varepsilon(u) H(\varphi) d\Omega \]

s.t. \[ \int_D \varepsilon^T(u) : C : \varepsilon(u) H(\varphi) dx = \int_D b_0 H(\varphi) dx + \int_{\Gamma_\tau} \tau v ds \quad \forall \ v \in U \]

\[ G(\varphi) = \int_D \frac{H(\varphi) d\Omega}{d\Omega} - V_{\text{max}} = 0 \]

(7)

where \( \Gamma_\varepsilon \) and \( \Gamma_\tau \) are Dirichlet and Neumann boundaries, \( J \) the objective function, \( u \) the displacement field, \( \varepsilon \) the linearized strain tensor, \( C \) the elasticity tensor, \( v \) is the adjoint displacement in the space \( U \) of the kinematically admissible displacement fields, \( b \) represents the body force, \( G(\varphi) \) is the volume constraint to limit material usage and \( V_{\text{max}} \) is the maximum allowable volume fraction of the design domain \( D \). \( u \), \( \tau \) are the given displacement, the boundary unit normal vector and traction respectively. \( H(\varphi) \) is the Heaviside function which is defined as:

\[ H(\varphi) = \begin{cases} 0 & \varphi < 0 \\ 1 & \varphi \geq 0 \end{cases} \]

(8)

In the problem of the compliance minimization without considering boundary tractions, the sensitivity analysis based on shape derivative can be used to derive the normal velocity \( V_e \) along the moving boundaries in the steepest descent direction, which can be calculated as [4]:

\[ V_e = \frac{1}{2} \varepsilon(u) : C : \varepsilon(u) - b \cdot u - \lambda \]

(9)

where \( \lambda \) is the Lagrange multiplier to control the volume constraint and it can be calculated with augmented Lagrange multiplier method or bi-section method. In this study, the velocity \( V_e \) is extended to the whole design domain instead of just being calculated along the boundary. With the velocity \( V_e \), the level-set function can be updated until converges to realize the optimal solution. Please note the sensitivity of the objective function may no longer be monotonic due to the presence of self-weight.

![Figure 1. A topology optimization problem under self-weight](image-url)
3. Numerical Examples

As shown in Figure 1, a rectangle plate ($L \times W$) is fixed at its left and right bottom ends. The boundary conditions and specific parameters of the structure are shown in Figure 1. The objective of the example is to determine the optimal topology with minimal compliance withstanding its self-weight. The length $L=0.80\text{m}$, width $W=0.40\text{m}$, thickness $t=0.01\text{m}$. 

3.1 Topology Optimization Results Using the Parameterized Level-Set Method

Physical material parameters are provided: Young’s modulus $E = 2.1 \times 10^{11} \text{Pa}$, density $\rho = 7850 \text{kg/m}^3$, and Poisson’s ratio $\nu = 0.3$. The whole structure is only affected by the gravity load. The gravitational acceleration $g=9.8 \text{m/s}^2$, the finite element analysis uses the bilinear quadrilateral elements, and the structure is divided into a Cartesian grid of $40 \times 80$, and the pseudo time step is $\Delta t=0.1$. The constraint is the volume of material usage, and the volume fraction is fixed at 0.5. $E_{\text{min}}=E \times 10^{-1} \text{Pa}, \rho_{\text{min}}=\rho \times 10^{-1} \text{kg/m}^3$.

![Figure 2. The intermediate solutions of the topology optimization process](image)

The optimization iteration process is shown in Figure 2 and the convergence curves of the objective function and the volume constraint are shown in Figure 3. As can be seen from Figure 2, the final optimization design is like two pillars connected by an arch in between. Based on the stiffness criterion, the optimal distribution of the material should be above the fixed points to form two pillars to reduce the bending moment (this can also be seen from a later example in this paper). However, the two pillars will tilt down to each other, because the single-point-fixed bottom of the two pillars cannot resist bending moment, the middle arch appears to prevent the pillars on both sides from tipping over to the middle. The optimal configuration is quite close to the result obtained by the BESO method [8], but it is different from the solutions obtained using the SIMP method, which will be seen in the following section.

3.2 Comparison of Results Between Several Methods

In this part, the optimal results obtained using the SIMP method, RAMP method, and Abaqus software are discussed and compared with the results obtained using the parameterized level-set method. The SIMP model is one of the most commonly used material interpolation models in variable density method. Its material interpolation function is

$$E_i = a_i^{\rho} E_{\text{iso}}$$

where $E_i$ is Young’s modulus of the element $i$ and $E_{\text{iso}}$ are Young’s modulus in the solid state, $\rho$ the penalty coefficient, $a_i$ the density of element $i$. Eq. (10) provides a power function between the Young’s modulus of the material and the pseudo-density $a_i$. According to that, the ratio between the gravity and Young’s modulus of the material of a unit volume is

$$\frac{a_i}{E(a_i)} = a_i^{1-\rho} \frac{1}{E_{\text{iso}}}.$$  

Eq. (11) is illustrated in Figure 5 with the dashed line marked with triangles when $\rho=3$. It can be seen when the pseudo-density $a_i$ approaching zero, the ratio becomes infinity large because the decrease speed of Young’s modulus is much faster than the decrease speed of the gravity due to the existence of the exponential relationship. That means, for the elements having very low densities, the weak material cannot withstand its own gravity, resulting in the excessive displacement of the corresponding nodes. In that case, the optimization process may not be stably
converged, and a typical unconverted solution is shown in Figure 4(a).

To overcome this drawback, the interpolation model can be improved by introducing a piecewise function [12];

$$E(a_r) = \begin{cases} a_l E_0 & \text{if } a_r \geq a_l \\ \frac{a_l}{a_l} x, E_0 & \text{if } a_l < a_r \end{cases}$$

(12)

$$\frac{a_l}{a_l} = \begin{cases} a_l^{1-p} \frac{1}{E_0} & \text{if } a_l \geq a_l \\ \frac{a_l^{1-p}}{E_0} & \text{if } a_l < a_l \end{cases}$$

(13)

where $a_l$ is a threshold with a small value. The ratio of Eq. (11) becomes

This improved model, which introduces a threshold $a_l$ to eliminate the infinite proportional effect in the original model, can successfully converge to a stable solution.

The RAMP method is another popular variable density method, in which the intermediate density is penalized to 0 or 1 by a proportional manner instead of the power function [12]. In this model, the interpolation function of the material property is defined as [13]:

$$E(a_r) = \frac{a_l}{1 + R(1 - a_l)} E_0$$

(14)

where $E(a_r)$ is the elastic modulus after interpolation and $R$ is a factor to adjust the penalty effect. The ratio of the mass to the stiffness can be represented as:

$$\frac{\rho'(a_r)}{E(a_r)} = \frac{a_l \rho'}{a_l} = \frac{[1 + R(1 - a_l)] \rho'}{E_0}$$

(15)

where $\rho'$ is the density of solid material. It can be seen that when the density approaches zero, the ratio in Eq. (15) of the RAMP model is still bounded, which can keep the optimization process stable.

![Figure 4](image)

Figure 4. The optimal results using several different algorithms

Figure 4 illustrates the optimal results of the problem defined in Figure 1 using several different algorithms. In Figure 4(a), even though the unconverted solution may be unreliable, there are many catenary and dendriform structures attached to the arch, which confirms some natural force phenomena. This method is expected to provide more convincing optimization results in future research. Figure 4(b) is a black-and-white optimization result in SIMP model with MMA. Figure 4(c) provides the solution obtained in Abaqus with SIMP model and Figure 4(d) is the result of the RAMP model. The solutions of the parameterized level-set method are given in Figure 4(e), which shows a little difference with the solutions of SIMP and RAMP models by having two bulges at the two sides on the top. This special shape can be verified by the lowest objective value in Figure 4 and the similar solution obtained using the
BESO method [8].

In addition to the above mentioned methods, inspired by Yang et al. [8] in applying the BESO model, the stress distribution can also be considered as a criterion to optimize the design by using the level of Von-Mises stress as the normal velocity in the parameterized level-set model. In the BESO model, the criterion for removing elements is the stress distribution, while in the parameterized level-set model, the normal velocity can be calculated as follows for a plane stress problem:

\[ V_n = \sqrt{\sigma_x^2 + \sigma_y^2 - 2\sigma_x\sigma_y + 3\tau_{xy} - \lambda} \]  

(16)

where \( \sigma_x \) and \( \sigma_y \) are the stress components in the \( x \) and \( y \) direction respectively, and \( \tau_{xy} \) the shear stress. The optimization is carried out under the same conditions and parameters, and it is interesting that the similar optimal shape with the SIMP method (Figure 4(a)) is obtained; a typical arch structure which is shown in Figure 4(d).

3.3 Topology Optimization Under Unconstrained Material Usage by the Parameterized Level-Set Method

Consider a rectangular two-dimensional structure optimization problem without the volume constraint (the boundary conditions and parameters are the same as in section 3.1). Figure 5 shows, around step 60, the structure breaks from the middle, forming two separate columns, and the strain energy has a dramatic change. With further reduction in material usage, the optimization result becomes no physical material left and the objective function approaches zero in the end. The optimization iteration process is illustrated in Figure 5.

The "empty structure" is indeed the theoretical optimal solution for the problem without volume constraint, because the objective function gets the minimal value; zero. In this paper, the parameterized level-set method obtains the blank solution which proves the statement and also verifies the effectiveness of this method.

4. Conclusions

This paper briefly introduces the research progress and current situation of the self-weight in topology optimization and compares the optimization results of the parameterized level-set model with several density-based optimization algorithms. The proposed method can well avoid the difficulty of the parasitic effect of low densities after adding the self-weight. It is found that the optimal configuration using the parameterized level-set method, with the lowest objective value, is different with the solutions of the SIMP and RAMP methods but is close to the shape obtained with the BESO method in the literature. In addition, the optimal shape based on the Von Mises stress criterion is similar to the SIMP method but the reason is still not clear yet. Meanwhile, without considering the volume constraint, the parameterized level-set method can converge to an empty structure, which can be verified as the global optimal solution. In summary, the parameterized level-set optimization method can effectively solve the stiffness optimization problem under self-weight.

Acknowledgments

This research was supported by the National Natural Science Foundation of China under Grant Nos. 11372004.
References

POROUS STRUCTURAL DESIGN FOR ADDITIVE MANUFACTURING
AND IMPROVED DAMAGE TOLERANCE

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Abstract
Porous structural design has attracted tremendous interest because of the advances in additive manufacturing and the superior characteristics of porous structures. This study presents a filter approach for porous structural design in the framework of single scale density based topology optimization. It in principle follows the idea to control porosity by limiting the local volume fraction of material. These local limits are usually incorporated into the optimization problem as a number of individual constraints or one condensed constraint. In contrast to the conventional constraint approach, a filter approach is proposed here to explicitly integrate these local limits into the material interpolation and have them automatically satisfied during the optimization iterations. A variety of numerical examples have been studied to validate the effectiveness of the filter approach.

Key words: Topology Optimization, Porous Structure, Additive Manufacturing, Damage Tolerance, Porosity Control

1. Introduction

Because of the advances in additive manufacturing, porous infill design is targeted as one of the crucial issues [1], and porous structural design has received an increasing number of studies in the community of topology optimization. Particularly, two studies intentionally exploited porosity to improve buckling strength [2] and damage tolerance [3]. While both of these two studies relate to structural safety, the work in [3] is closely related to fail-safe design [4-5], which means the structure can still resist the design load when a local failure occurs. Compared with fail-safe design, porous structural design needs much less computation and therefore can potentially be used as a convenient way to improve fail-safe behaviour.

The proposed porous structural design method falls into the category of single scale topology optimization, i.e., without separation of macro-scale structure and micro-scale material. The major advantage of single-scale optimization is that the whole porous structure is analyzed and optimized in one macroscopic scale, and thus it does not suffer from connection problem among different types of units encountered in multi-scale topology optimization. Furthermore, its numerical implementation is simple, since researchers in the community of topology optimization are familiar with single-scale topology optimization. For porous structural design, the major disadvantage of single-scale topology optimization is that its computation can be more expensive than multi-scale topology optimization. However, the required computing resources are available today. Solving a single-scale topology optimization problem for porous structural design with course and moderate resolution can be achieved on PC and high-resolution designs can be achieved with GPU [6] or cluster [7-8].

In single-scale topology optimization, several studies have been made to generate optimized porous structures based on a variety of approaches, for example, the SIMP method [8-9], the Level Set Method (LSM) [10-11], the Bi-directional Evolutionary Structural Optimization (BESO) method [12] and Voronoi diagram based method [13-15]. The study in this paper is based on the SIMP method. Alexander and Lazarus [9] first applied a single-scale optimization method to analyze and optimize porous structures a whole body without separation of macro- and micro-scale. However, the porous structures in their study were still assumed to be periodic and layered. The authors claim that micro-structural details can vary in all directions when applying a number of local limits of material as studied in [16] for maximal length scale control. This was investigated by Wu et al. [3], where they applied single-scale topology optimization to design optimized porous infill structures approaching bone like structures by using one condensed constraint based on [16]. The constraint approach in [16] was also adopted by Zhao et al. [12] to study
biomechanical morphogenesis of the leaves and stalks of representative emergent plants based on the BESO method. These related studies have mainly focused on the constraint approach to control porosity. Just like overhang angle control can be realized by using both the constraint approach [17] and filter approach [18-19], it should also be possible to control porosity by using filter approach. A filter approach to control porosity not only provides an alternative to the constraint approach, but also opens new opportunities for future studies such as a potential combination with other filter approaches for additive manufacturing [18-19]. The aim of this paper is to provide a novel filter approach for porous structural design. The idea is to integrate these local limits into the material interpolation and have them automatically satisfied in the optimization.

The paper is organized as follows. Section 2 introduces the filter approach. Section 3 presents the results of a benchmark example. Section 4 draws the conclusions.

2. Method

2.1 Formulation of the Optimization Problem

Firstly the standard minimum compliance problem is considered as follows

\[ \min_{x} c(x) = u^T K u \]

subject to: \[ K(x) u = f \]
\[ V(x)/V^* \leq \alpha \]
\[ 0 \leq x \leq 1. \]  

(1)

Where \( x \) is a vector of design variables, \( c \) denotes the compliance, \( K \) is the global stiffness matrix, \( u \) and \( f \) are the global displacement and force vector, respectively, \( V \) and \( V^* \) are the material volume and design domain volume, respectively, and \( \alpha \) is the prescribed volume fraction. The conventional optimization based on Eq. (1) does not necessarily produce desirable porous structures, though porous structures can appear in the results in case of large-scale topology optimization.

For porous structural design, the local percentage of material can be limited by imposing a number of local constraints. In a straightforward way, the optimization problem for porous structural design can be formulated as

\[ \min_{x} c(x) = u^T K u \]

subject to: \[ K(x) u = f \]
\[ V_e(x)/V_e^* \leq \eta, \forall e \]
\[ V(x)/V^* \leq \alpha, \text{optional} \]
\[ 0 \leq x \leq 1. \]  

(2)

where \( V_e \) and \( V_e^* \) are local material volume and local design domain volume measured in a prescribed neighborhood of each element \( e \), respectively, and \( \eta \) is the prescribed limit of local volume fraction of material. For porous structural design, the global volume constraint becomes optional, since the total material volume is already limited by the local volume constraints. For the sake of computational efficiency, the local constraints can be aggregated into one condensed constraint. This constraint approach was studied in [3] and [12] for porous structural design. The local constraints in Eq. (2) and their variants are referred to as constraint approaches for porosity control.

In contrast to the above constraint approaches, a filter approach for porous structural design is presented in this paper. For the filter approach, a major change to the optimization problem in Eq. (2) is that the local constraints are removed. The resulting optimization problem looks the same as Eq. (1) and therefore is not re-written here for brevity. It is emphasized that the global volume constraint in this case is also optional as discussed in Eq. (2). After removing these local constraints from the optimization problem, their effects are retained through a novel filter approach described as follows.

2.2 Filter Approach for Porous Structural Design

Previous studies have presented a number of filter approaches for different purposes, including minimal length scale control, maximal length scale control and overhang angle control. Here, the purpose of the new filter approach is to
introduce and control porosity of the optimized design.

A flow diagram of the proposed filter approach is illustrated in Figure 1. It consists of a standard filtering and projection step, an additional filtering and projection step, and one multiplication. The idea is that one element can become solid when it wants to become solid without porosity control, and it satisfies the local limit in case of porosity control. For the sake of an intuitive physical meaning, a uniform filter is currently used in the second filtering to accurately calculate the local percentage of material. It is noted that it is possible to use other types of filters such as PDE filter in the second filtering process. In the second projection, a so-called low-pass projection function is used. In a discrete form, it projects the scalar value below a prescribed threshold to one and the scalar value above the prescribed threshold to zero. In the numerical implementation, a smoothed form is used, which is defined based on the smoothed Heaviside step function. The threshold parameter in the second projection is chosen based on the prescribed limit of the local percentage of material.

Figure 2 illustrates the function of the proposed filter approach. For a given structural feature (shown as solid), the filter approach can open a hole (shown as void) in the center of structural feature where the local volume fraction of material exceeds a prescribed limit, for example, 50%. It is seen that the filter approach can effectively modulate the design space such that porosity is introduced into the structure. Hence, the proposed filter approach is called porosity control filter or porosity filter in terms of its function.

![Design variable](image1)

**Figure 1.** A flow diagram of the proposed density filter approach for porosity control

![Illustration](image2)

**Figure 2.** Illustration of the proposed filter approach. A: the scalar field resulting from the first projection. B: the scalar field after applying the second filter. C: the scalar field after applying the second projection. D: the final scalar field obtained by multiplying the two projected fields, i.e., A and C. Black and white colors denote solid and void, respectively.
3. Numerical Examples

To investigate the effectiveness of the proposed filter approach, a range of examples have been studied. For the sake of brevity, only a benchmark cantilever beam problem is reported here.

Figure 3A shows the definition of the structural optimization problem, including design domain, boundary condition and external load. The objective is to find an optimized porous structure with minimum compliance. The design domain is discretized by using $400 \times 200$ elements for a coarse-resolution design and $800 \times 400$ elements for a moderate-resolution design. Figure 3B and 3C display two representative designs obtained by using the proposed filter approach. These results show that the proposed filter approach is effective to obtain optimized porous structures. Figure 4 shows the evolution of the optimized design during optimization iterations for Figure 3C. It shows that the design converges to a black-and-white design in a smooth manner.

![Figure 3](image3.png)

**Figure 3.** Porous structural design of a cantilever beam. A: definition of the design domain, boundary condition, and external load. B and C: two representative designs obtained by using the proposed filter approach to control porosity.

![Figure 4](image4.png)

**Figure 4.** Evolution of the optimized design during optimization iterations. The designs are consecutively numbered.

4. Conclusions

This study set out to investigate a filter approach for porous structural design in the framework of single-scale density based topology optimization. The numerical examples suggested the proposed filter approach can effectively control...
the porosity of the resulting structure. In future work, the porosity control filter proposed here can be combined with other filter approaches such as overhang angle control filter [18-19] to generate optimized self-supporting porous structures.

**Acknowledgments**

This work is supported by the research project SELMA: Fail-safe Structural Optimization funded by Independent Research Fund Denmark.

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AN OVERHANG CONSTRAINT ADAPTABLE TO A PROPER BUILDING ORIENTATION

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Abstract

In additive manufacturing processes, the critical overhang angle of downward facing surfaces limits the printability of parts. To consider this limitation of the process in topology optimization, several approaches have been proposed in the literature. Most of them operate with a user-defined building direction, thus, if the orientation is not appropriate, structural performance could be drastically compromised. This work aims to reduce the dependence of the user on the definition of the building direction. We make use of a gradient-based constraint due to the low computation cost it demands in comparison to layer-by-layer approaches. The method is demonstrated on 2D and 3D examples.

Key words: SIMP, Manufacturing Constraints, Aggregation Functions

1. Introduction

Additive Manufacturing (AM) and topology optimization have allowed significant progress in the development of highly efficient components. However, technological limitations prevent the direct link between these technologies due to the post-processing labor of the optimized parts. This task is often necessary to meet the manufacturing constraints of AM technologies. Aimed at producing print-ready designs, several authors have recently proposed methods to include the overhang limitation of AM in the topology optimization problem. These methods can be classified into two groups: the gradient-based [1-2] and the layer-by-layer-based approaches [3-4]. The first one has the advantage of implementation, since it evaluates the overhang condition using gradient-based information to detect the inclination of the parts, being easier to implement and requiring less computational cost than the layer-by-layer approaches. The local nature of gradient-based approaches, however, promotes the appearance of triangular features that meet the local constraint but are not self-supporting, as shown in Figure 1b. Even though it is possible to obtain results without this undesirable feature at the expense of significant tweaking of parameters or by adding additional constraints [1]. We have noted, however, that this type of formulations offer good performance in a few printing directions, which usually coheres with the direction that has the least downward-facing surface (see Figure 1(c)). Inspired in this observation and in the work of Langelaar [5], in this work we propose an overhang constraint that aims to follow the best printing direction during the optimization process. The development is demonstrated in the robust design context [6], therefore, solutions reported in this document include two-phase minimum length scale control.

![Figure 1. The MBF beam for compliance minimization with overhang constraint [1]. In (a) the reference solution without overhang constraint. The constraint considering a vertical (a) and horizontal (b) building direction.](image)

2. The Utilized Overhang Constraint

Due to its low computational cost, easy implementation and parallelizable structure, we have chosen the constraint proposed by Qian [1] as a basis. Unlike the original contribution [1], in this work we do not use the grayness constraint, and the local overhanging angle restrictions are aggregated in a different way, which is described below.
The local constraint \( g_i \), i.e., the constraint applied on each element to control the overhanging angle, is defined in (1). There, \( \mathbf{b} \) is a vector with components \([ b_x, b_y ]\) that defines the printing direction; \( \mathbf{v}_a \) the gradient of a user-defined reference surface; and \( \mathbf{\bar{d}} \) the normalized gradient vector whose components are \([ \mathbf{\bar{d}}_{x,i}, \mathbf{\bar{d}}_{y,j} ]\). The restriction (1) asks the projection of the gradient \( \mathbf{\bar{d}} \) in the printing direction \( \mathbf{b} \) be less than the projection of the reference surface \( \mathbf{v}_a \), as shown in Figure 2. There it can be observed that the printing direction has a significant effect on the condition of the restriction, where in Figure 2a \( g_i \) is satisfied but in Figure 2b \( g_i \) is violated.

\[
g_i = \mathbf{\bar{d}} \cdot \mathbf{b} - \mathbf{b}^T \cdot \mathbf{v}_a \leq 0
\]  

(1)

Figure 2. Gradient-based overhang constraint \([1]\). In (a) for a vertical building orientation where the local constraint is satisfied. (b) For an inclined building orientation where the overhang constraint (1) is violated.

In 2D, the gradient vector \( \mathbf{\bar{d}} = [ \mathbf{\bar{d}}_{x,i}, \mathbf{\bar{d}}_{y,j} ] \) is computed as (2), (3), being \( \mathbf{D}_x \) and \( \mathbf{D}_y \) matrices that assemble the weights to obtain the discrete gradient. As an approximation of the gradient we use the Prewitt operator, which is a regular domain considers unit weights, as shown in Figure 3 for the 2D and 3D case. Thus, the components of the \( \mathbf{D}_x \) matrix are defined in (4), where \( x_i \) and \( y_j \) are the \( x \) and \( y \) coordinates of the centroid of the element \( i,j \); \( h_x \) is the size of the element; and \( \mathbf{\hat{p}} \) is the projected field. The components of the \( \mathbf{D}_y \) matrix are the same as \( \mathbf{D}_x \), but using the \( y \)-coordinates. The normalized gradient vector \( \mathbf{\bar{d}} \) is obtained from \( \mathbf{\bar{d}} \) as \( \mathbf{\bar{d}}_{x,i} = \mathbf{d}_{x,i} / \mathbf{\bar{d}} \) and \( \mathbf{\bar{d}}_{y,j} = \mathbf{\bar{d}}_{y,j} / \mathbf{\bar{d}} \).

\[
\begin{align*}
\mathbf{d}_{x,i,j} &= \begin{cases} 
\frac{x_j - x_i}{|x_j - x_i|} & \text{if } 0 < |x_j - x_i| \leq h_x, \\
0 & \text{otherwise}
\end{cases} \\
\mathbf{\bar{d}}_{x,i} &= \mathbf{D}_x \mathbf{\hat{p}} \\
\mathbf{\bar{d}}_{y,j} &= \mathbf{D}_y \mathbf{\hat{p}}
\end{align*}
\]

(2)

Figure 3. Prewitt weights used to approximate the gradient operator in the \( x \)-direction. (a) for 2D and (b) for 3D.

The local constraint is aggregated into a single one using the \( p \)-mean function. To do this, the \( g_i \) constraint is shifted as in (5) to remain in the domain \([0, 1]\). The global restriction \( G \) is expressed in (6), being \( p \) the aggregation parameter.

\[
g_i = 0.5 \left( g_i + \mathbf{b}^T \cdot \mathbf{v}_a \right) + 1
\]

(5)

\[
G = 2 \left( N^{1/p} \sum_{i=1}^{N} (s_i)^{1/p} - (1 + \mathbf{b}^T \cdot \mathbf{v}_a) \right) \leq 0
\]

(6)

3. Considering the Building Direction

The problem with the overhanging angle constraint (6) is that the printing direction \( \mathbf{b} \) must be selected. This manufacturing criterion is usually left free to the user, which can greatly limit structural performance if the direction
is not well chosen. Langelaar [5] proposes to evaluate several printing directions and choose the least penalizing one through aggregation functions. The same criterion is applied here, but in a formulation that is easily parallelizable due to the absence of the layer-by-layer analysis.

In this work, the same as [5], the user must propose m printing directions, therefore m overhanging angle constraints (7) must be computed, i.e., $G_k$, $k = 1, \ldots, m$. The strategy used here is to aggregate the m constraints $G_k$ using a r-mean function to capture the least penalizing function, i.e., $\min (G_k)$. To this end, the quantity of interest that is aggregated is (8) and the global constraint $B$ which considers m candidate directions is in (8),

$$t_i = 1 - 0.5 \left( G_{\alpha, a} + b_i \cdot 8v_a + 1 \right) \quad (7)$$

$$B = 1 - 2 \left( m^{-1} \sum_{i=1}^{m} (t_i)^{-r} - b^T \cdot 8v_a \right) \leq 0 \quad (8)$$

In this work we use the robust design approach [6], which uses 3 different projections: erode $\tilde{p}^m$, intermediate $\tilde{p}^r$, and dilate $\tilde{p}^d$. These are obtained from the filtered field $\tilde{p}$ using the Heaviside function shown in (9), where $\beta$ and $\mu$ control the steepness and threshold of the projection, respectively. To obtain $[\tilde{p}^m, \tilde{p}^r, \tilde{p}^d]$ in this work we use $\mu = [0.75, 0.50, 0.25]$. Bearing in mind that we use the density filter with a filter radius $r_{\text{fit}}$, the theoretical minimum size obtained in the physical field is $r_{\text{min}} = 0.5r_{\text{fit}}$, for both 2D and 3D problems.

$$\tilde{p}(i) = \frac{\tanh(\beta \mu) - \tanh(\beta - (\beta - \mu))}{\tanh(\beta \mu) + \tanh(\beta - (1 - \mu))} \quad (9)$$

We solve the compliance minimization problem subject to volume and overhanging angle constraints (10). The compliance is computed for the erode field only since it is the least stiff (see [7] for more details). The overhanging angle constraint is evaluated in the erode and dilate fields only, since in most iterations of the optimization process, $B_{\alpha}(\tilde{p}^d)$ is in between $B_{\alpha}(\tilde{p}^r)$ and $B_{\alpha}(\tilde{p}^d)$.

$$\begin{align*}
\text{min.} & \quad f^T u \\
\text{s.t.} & \quad K_{uv} u = f \\
& \quad v^T \tilde{p} \leq V \\
& \quad B(\tilde{p}) \leq 0 \\
& \quad 0 \leq \rho \leq 1 
\end{align*} \quad (10)$$

**Sensitivity Analysis**

By applying the chain rule, and considering that design variables are stored in column arrays, the derivative of the constraint $B$ with respect to the design variables are as follows:

$$\frac{dG_k}{dp} = \frac{
abla \tilde{G}(\tilde{p})}{
\partial \tilde{p}} = \frac{\left( \frac{d \delta}{dp} \frac{d \delta}{dp} + \frac{d \delta}{dp} \frac{d \delta}{dp} \right) \nabla \tilde{G}(b)}{\delta} + \frac{\left( \frac{d \delta}{dp} \frac{d \delta}{dp} + \frac{d \delta}{dp} \frac{d \delta}{dp} \right) \nabla \tilde{G}(b)}{\delta}$$

$$\frac{dB}{dp} = \frac{dG}{dp} \frac{dG}{dp} \frac{dB}{dp}$$

The additional computational cost is proportional to the number of candidate directions $m$ because of the amount of constraints (8) and derivatives (11) that must be computed in each iteration. However, some strategies can be implemented to reduce the cost when a large amount of directions are considered. For instance, it is possible to use an active set approach to remove some fraction of candidate directions, which reduces the computations in (11). Another way, which is implemented in this work, is to store the arrays that appear in (11) and do not depend on $b$. In addition, after some arrangements of the expression (11), the Hadamard product can be used to speed up the computation. The substitution of the derivatives in (11) and (12) will be omitted to avoid overextending the content of the article.

**4. Examples**

We solve the MBB-beam for compliance minimization as a test case. The design domains are shown in Figure 4. The modified SIMP interpolation scheme [8] is used where $\eta$ is the penalization parameter. A continuation procedure is used to define $\eta$ and $\beta$, which ends with $\eta=3$ and $\beta=25$. The volume constraints are 40% and 20% for the 2D and
3D case, respectively. The constraint is implemented in the PolyTop and TopOpt codes [9-10] to solve the 2D and 3D problems, respectively. MMA [11] is used as an optimizer and the remaining optimization parameters are kept as in the original codes [9-10].

The building direction \( \mathbf{b} \) is defined in polar coordinates, where \( \theta \) is the polar angle defined in Figure 4(a). For 3D, spherical coordinates are used, where \( \theta \) and \( \varphi \) are the polar and azimuthal angles, as shown in Figure 4(b). Since we do not include the support domain outside the design space, when the printing direction is inclined with respect to the boundaries of the design domain, the closer surface to the “bottom” of the design domain serves as baseplate, as shown in Figure 4(c). In all examples we use \( \rho=60^\circ, r=10 \) and \( 45^\circ \) as overhanging angle limit.

![Figure 4](image)

Figure 4. In (a) and (b), the MBB beam design domain and the reference angles to define the building orientation. In (c), the surface of the design domain serves as baseplate when inclined building orientations are considered.

Figure 1(a) in the Introduction shows the reference solution. Figure 1(c) corresponds to the solution obtained with 4 candidate directions, \( \theta=[0^\circ, 90^\circ, 180^\circ, 270^\circ] \), where \( \theta=180^\circ \) is the preferred one by the utilized method. Figure 5(a) shows the solution obtained with 36 candidate directions, starting from \( \theta=0^\circ \) up to \( \theta=370^\circ \), with \( \Delta \theta=10^\circ \). In this case, the prevailing direction is \( \theta=170^\circ \), thus the printing would be carried out as depicted in Figure 4(c). It is interesting to note that the solution obtained with \( \theta=180^\circ \) in Figure 1(c) does not contain the diagonal bar labeled as \( d_1 \) due to the overhanging angle restriction. But when building orientations increase, as when using \( m=36 \), the \( d_1 \) bar remains in the design. It can be seen that the differences between the reference solution and the one that considers \( m=36 \) are subtle. The last one presents small modifications that allow the structure to be self-supporting. This is one of the main features sought by designers in the post-processing stage, i.e., the minor material redistribution to facilitate the printability of the structure. Figure 5(b) shows the solution obtained using an overhanging angle limit of \( 60^\circ \). In this case the preferred printing direction is \( \theta=180^\circ \). This suggests that the optimal printing direction depends not only on the design space, but also on the overhanging angle limit.

![Figure 5](image)

Figure 5. MBB-beam for compliance minimization with overhanging angle constraint of (a) \( 45^\circ \) and (d) \( 60^\circ \). In (b) and (c) the global constraints \( G_c \) obtained for each candidate direction. In (c) and (f) the constraint map on the blueprint design \( \bar{G}_c(\bar{\rho}) \).

The solutions of the 3D-MBB are shown in Figure 6. The reference solution that does not include the overhang constraint is shown in Figure 6a. In this example, 2 sets of candidate directions are used. One problem includes 4 candidate directions were \( \theta=[0^\circ, 90^\circ, 180^\circ, 270^\circ] \) and \( \varphi=0^\circ \). The second one includes 72 candidate directions with \( \theta \).
\(=\left[0^{\circ} ; 10^{\circ} ; 270^{\circ}\right]\) and \(\varphi=\left[0^{\circ}, 90^{\circ}\right]\). In both cases, the preferred printing orientation is \(\theta=180^{\circ}\) and \(\varphi=0^{\circ}\). The constrained solution is shown in Figure 6(b). It can be observed slight differences between the two designs. The main one is located in the upper central part of the structure, which is highlighted with red arrows. The shadow in the bottom of the structure offers a clearer sight of it. As expected, the proposed restriction chose the least penalizing direction and rearranged the material to make the structure self-supporting.

![Diagram](image)

Figure 6. 3D-MBB beam. In (a) the solution without and (b) with overhang constraint.

5. Conclusions

A gradient-based overhang constraint was used in this work to evaluate printability for different building orientations. The least penalizing constraint is added in the topology optimization problem using aggregation functions. The MBB-beam for compliance minimization was used as test case. It was observed that the printing direction has an influence on the structural performance, so it should not be left as an extra variable to the designer. A greater effect is observed in the resulting topology, where the building orientation may modify the number of bars or holes that define the structure.

Due to the local nature of the gradient-based overhang constraint, the method is easily parallelizable and of low computational cost. This allows the evaluation in different printing directions with a low computational cost if a low number of candidate directions are considered. However, despite using the robust design approach, the constraint does not guarantee self-supporting structures since it evaluates the slope of the surface and not the AM process. However, satisfactory performance can be expected from gradient-based constraints in design domains that promote diagonal bars, such as the MBB beam, the cantilever beam, or the force inverter.

Acknowledgments

The work was supported by the project AERO++, funded by the Plan Marshall 4, 0 and the Walloon Region of Belgium. Computational resources have been provided by the Consortium des Équipements de Calcul Intensif (CECI), funded by the Scientific Research Fund of Belgium (F.R.S.-FNRS).

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TOPOLOGY OPTIMIZATION OF FLEXIBLE MULTIBODY SYSTEMS USING A PARAMETRIZED LEVEL SET METHOD

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Abstract

A procedure for topology optimization of flexible components in nonlinear dynamic multibody systems is propounded. Thereby, the bodies are modeled by the floating frame of reference approach. The optimization is based on a parametrized level set method. In this procedure, the dynamic response of the flexible components is replaced by a set of equivalent static loads, from which an objective function and its sensitivity information, namely the normal velocity of the optimization updating scheme, are computed. In order to maintain the robustness of the optimization, the flexible components are parametrized by a SIMP-method and a control sequence is included to avoid instability due to an inaccurately selected normal velocity. The procedure is applied to optimize the flexible rod of a slider-crank mechanism, whereby two different normal velocities are utilized and the results are compared with each other.

Key words: Topology Optimization, Flexible Multibody Systems, Floating Frame of Reference Approach, Parametrized Level Set Method, Equivalent Static Load Method

1. Introduction

In the new generations of dynamical systems, the application of light-weight components is inevitable, which has its own advantages and drawbacks. Considering the concepts of light-weight construction, the material and energy consumption of the developed mechanisms can be reduced. However, these constructions undergo in general large nonlinear motions including vibrations and deformations. These undesirable deformations have to be considered in the modeling and simulation. For this purpose, the well-established method of flexible multibody systems can be used, see [9, 12]. Assuming that the deformations of the flexible bodies remain small and elastic, the floating frame of reference approach can be utilized to model the flexible multibody system, see [9, 12]. In this formulation, global shape functions obtained by model order reduction from FE-models are used to approximate the body deformations, see [4] for details. By a careful choice of the set of global shape functions, the efficient modeling and simulation of the flexible multibody system are guaranteed.

In the case of flexible multibody systems, the topology optimization is more complex than for conventional static problems. This is due to the fact that the governing equations of the flexible multibody systems are described by nonlinear differential equations. Furthermore, the loads on the flexible components depend on the time, the state and design variables. In prior topology and shape optimizations of flexible multibody systems, the well-known solid isotropic material with penalization (SIMP) method is mostly used to parametrize the underlying FE-model and the optimization algorithm is based on the Method of Moving Asymptotes, see for instance [4]. However, there are several first works, where different level set methods (LSM) are utilized in dynamic models, see [10-11]. In this work, a parametrized level set method, see [14-15], is applied for the topology optimization of flexible multibody systems, which are modeled using the floating frame of reference approach. Thereby, the utilized model order reduction allows for an accurate topology optimization on fine FE-meshes without increasing the computational effort of the multibody simulation.

The remainder of this paper is organized as follows. Section 2 introduces briefly the floating frame of reference approach. In section 3, the main idea of the applied parametrized level set method is shortly described. Then, in section 4, the optimization problem is formulated and different approaches for obtaining an appropriate normal velocity, which serves as sensitivity information of the optimization procedure, are addressed. In section 5, the
application example, namely a flexible slider-crank mechanism, is introduced, and the results of the optimization procedure are presented. After all, section 6 concludes with a brief summary of the main issues in this work.

2. The Floating Frame of Reference Approach

In the floating frame of reference method, the deformation of a flexible body, which is assumed to be small and elastic, is described in a reference frame \( K_R \). The large nonlinear rigid body motion is captured as the relative movement between the reference frame \( K_R \) and the inertial frame \( K_I \). This subdivision is depicted in Figure 1. Using a global Ritz approach, the elastic displacement \( u \) and the rotation vector \( \omega \) of the coordinate system \( K_r \), which is rigidly attached to a point \( P \), can be approximated as

\[
\begin{align*}
\mathbf{u}_r(R_{0P}, t) & \approx \Phi(R_{0P})\mathbf{q}_r(t) \\
\mathbf{\omega}(R_{0P}, t) & \approx \Psi(R_{0P})\mathbf{q}_r(t)
\end{align*}
\]

In this expression, the vector \( \mathbf{q} \in \mathbb{R}^n \) of \( n \) elastic coordinates is time-dependent, whereas the matrix \( \Phi \in \mathbb{R}^{3 \times n} \) of \( n \) global displacement shape functions and the matrix \( \Psi \in \mathbb{R}^{3 \times n} \) of \( n \) global rotation shape functions depend only on the relative position \( R_{0P} \) of the point \( P \) in the reference configuration. The shape functions can, for instance, be obtained by performing a model order reduction of a linear FE-model of the flexible body, which reads as

\[
\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}.
\]

Thereby, \( \mathbf{M} \), \( \mathbf{D} \) and \( \mathbf{K} \) are the mass, damping and stiffness matrix, \( \mathbf{q} \in \mathbb{R}^n \) is the vector of \( n \) nodal coordinates and \( \mathbf{f} \) represents the vector of applied loads. In order to approximate the elastic behavior of flexible bodies with a high accuracy, usually a fine discretization of the FE-model is required, which leads to a great number of nodal coordinates. Different model reduction methods can then be used, see [4], to maintain the efficiency of the multibody simulation. In this work, the vector \( \mathbf{q} \) of nodal coordinates is projected onto the vector \( \mathbf{q}_r \) of elastic coordinates by applying the Component Mode Synthesis (CMS), see [3], where \( n_c \ll n \).

Considering the \( n \) elastic coordinates and in addition the rigid body coordinates as degrees of freedom, the equations of motion of a free flexible body can be formulated by applying, for instance, Jourdain’s principle, see [12]. In multibody systems, different elements connect rigid and flexible bodies with each other and their environment, and reduce, in this way, the number of degrees of freedom, as shown in Figure 2. Performing a coordinate partitioning, the equations of motion in minimal form can be derived as ordinary differential equations (ODEs) with the following general form

\[
\dot{\mathbf{y}} = \mathbf{Z}(\mathbf{y}, \mathbf{z})\mathbf{z}
\]

\[
\mathbf{M}(t, \mathbf{y}, \mathbf{z})\mathbf{\dot{z}} + \mathbf{\gamma}(t, \mathbf{y}, \mathbf{z}) = \mathbf{f}(t, \mathbf{y}, \mathbf{z}, \mathbf{z})
\]

Thereby, \( \mathbf{y} \) and \( \mathbf{z} \) are the generalized position and velocity coordinates, whereas \( \mathbf{y} \) and \( \mathbf{z} \) are the redundant position and velocity coordinates. In the kinematic equation (3), \( \mathbf{Z} \) is the kinematic matrix, which connects the first time derivative of generalized position coordinates \( \mathbf{y} \) to the redundant velocity coordinates \( \mathbf{z} \), and in the kinetic equation (4), \( \mathbf{M} \), \( \mathbf{\gamma} \) and \( \mathbf{f} \) represent the global mass matrix, the vector of generalized local accelerations and the generalized vector of applied forces, respectively.

![Figure 1. Floating frame of reference approach](image1.png)

![Figure 2. Flexible multibody system](image2.png)

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3. Level Set Method

The level set method had been originally utilized in the framework of a front propagation problem and enabled the moving boundaries to perform high topological complexities and changes, see [8]. This method has then been applied in various technical areas, among others, in the structural optimization, see for instance [1, 13]. The main idea of the LSM is to define a level set function $\phi$ over a design domain $D$ of desired dimension $d$ (here $d=2$). Depending on the value of $\phi$ at an arbitrary point $Q$ with the coordinates $x_Q = (x_Q, y_Q)$, this point is assigned to the interior, exterior or free boundaries of the design. For instance, in Figure 3, the level set function creates an elliptical hole at the center of a 2D beam.

![Figure 3. Definition (a) of a level set function $\phi$ over the design domain $D$ and the resulting design](image)

During an optimization procedure, the level set method can be updated with respect to the following Hamilton-Jacobi (HJ) partial differential equation, see also [8, 13],

$$\frac{\partial \phi}{\partial t} - V_s |\nabla \phi| = 0 \quad (5)$$

where $t$ is a pseudo-time, $V_s$ is the normal velocity and $\nabla \phi$ is the gradient of the level set function. In this work, a radial basis functions based level set method with an augmented re-initialization is applied, which is presented in detail in [14-15]. Thereby, Multi Quadratic (MQ) splines are used to interpolate the level set function $\phi$ on a discretized design domain with $m$ nodes. It follows

$$\phi(i) = G \mathbf{a}(i) \quad (6)$$

In this formulation, $\mathbf{a}(i) = [a_0(i), \ldots, a_m(i), p_0(i), p_1(i), p_2(i)]^T$ includes the expansion coefficients $a_i$ of $m$ nodes and in addition the coefficients $p_0(i)$, $p_1(i)$ and $p_2(i)$ of a polynomial, which is added to encompass the constant and linear portions of $\phi$ and ensures the positive definiteness of the solution, see [6]. Moreover, the time-independent collocation matrix $G$ results from the evaluation of the radial basis functions and the additional polynomial at the $m$ nodes, and the vector $\mathbf{a}(i) = [\phi(i), \ldots, \phi_m(i), 0, 0, 0]^T$ includes the value of the level set function $\phi$ at these nodes. Using the decoupled formulation (6), the HJ partial differential equation (5) can be transformed to a system of non-linear ODEs. Utilizing for instance the first-order forward Euler’s method, and considering some additional adaptations, the generalized expansion coefficients can be updated from time point $i_k$ to time point $i_{k+1}$ as

$$\mathbf{a}(i_{k+1}) = \hat{\mathbf{a}}(i_k) + \tau G^{-1} \hat{B}' \mathbf{a}(i_k), \quad (7)$$

Thereby, $\tau$ is the time step size, $\hat{\mathbf{a}}(i_k)$ is the adapted vector of expansion coefficients at $K$-th time point and $\hat{B}' \mathbf{a}(i_k), i_k = [V_s(x, y), \cdots, V_s(x, y), \cdots, V_s(x, y), \cdots, \delta(\phi(x, \mathbf{a}(i_k)))], \cdots, V_s(x, y), \cdots, \delta(\phi(x, \mathbf{a}(i_k)))]^T$ is the adapted velocity vector. Whereby $\hat{a}(\phi)$ is an approximate function and hinders a limitless magnification of the level set function. The updating scheme (7) is only consistent with the formulation (5) if the level set function is a signed distance function, i.e., $|\nabla \phi| = 1$. For this reason, in [15] and augmented re-initialization is applied, which keeps the gradient magnitude at least around the free boundaries in the range of $|\nabla \phi| = 1$. The idea is to adapt $\phi$ by dividing through the mean of gradient magnitudes around the zero level set. Based on the linear relationship between $\phi$ and $\mathbf{a}$, this change is directly assigned to $\mathbf{a}$ so that the adapted vector $\hat{\mathbf{a}}$ results. After each update, $\phi(i_{k+1})$ results by evaluating the system (6) of linear equations. In the presented scheme, the level set function is bounded and remains smooth, which helps to overcome numerical issues. Furthermore, the approximate re-initialization does not hinder a hole creation. Hence, this procedure is appropriate for accurate topology optimization.

4. Topology Optimization

In this work, the modified SIMP law presented in [7] is applied to the underlying FE-model (2). Thereby,
introducing a density-like parameter \( x_i(\phi) \in (0,1] \) for each element \( i \), the corresponding Young's modulus \( E_i \) and the density \( \rho_i \) are penalized as

\[
E_i = x_i(\phi)E, \quad \rho_i = \begin{cases} \rho e^{x_i(\phi)} & \text{for } x_{\text{min}} \leq x_i(\phi) < 0.1 \\ \rho & \text{for } 0.1 \leq x_i(\phi) \leq 1 \end{cases}
\]

where \( E \) and \( \rho \) are the Young's modulus and the density of the reference solid material. Moreover, \( \rho \), \( q \) and \( c \) represent the parametrization constants, and \( x_{\text{min}} \) is the lower parameter limit for \( x_i \), which is imposed to avoid numerical issues such as ill-conditioned system matrices. Here, the parameter limit is set to \( x_{\text{min}} = 0.01 \) and the penalization constants are selected as \( \rho = 3 \), \( q = 6 \) and \( c = 10^7 \). The penalization approach (8) has no influence on a completely filled element \( i \) with \( x_i(\phi) \), whereas the material behavior of intermediate elements around the boundaries is adapted. This change contributes to the robustness of the optimization procedure. In this work, the well-known compliance minimization problem is considered, which is formulated in the weakly coupled form. In other words, the optimization procedure is not directly based on the dynamic response, but on the equivalent static loads, see [5]. More precisely, the dynamic forces within a certain time interval \( [t_i, t_{i+1}] \) are replaced by a number \( l \) of equivalent static loads \( f^{\text{eq},j} \), with \( j = 1(1)l \), which lead to the same deformation of the elastic bodies as the dynamic forces at the \( l \) selected time points \( t_i \). The compliance minimization problem of each elastic body with \( n \) finite elements can then be written as

\[
\begin{align*}
\text{minimize} & \quad \phi(\phi) = \frac{1}{l} \sum_{i=1}^{l} \sum_{j=1}^{n} v(x_i, y_j(\phi) u_i^j \mathbf{K}_{i,j} \mathbf{u}_i^j, \\
\text{subject to} & \quad h(x_j^i, \phi) = V(\phi) - V_{\text{min}} \leq 0.
\end{align*}
\]

Thereby, \( \mathbf{u}_i^j \) represents the displacement vector of the \( i \)-th element at the \( j \)-th time point and \( \mathbf{K}_{i,j} \) is the stiffness matrix of the \( i \)-th element before penalization. Here, the objective function \( \phi \) of the design \( \phi \) is expressed as the sum of compliances of \( n \) elements at \( l \) time points divided by the number \( l \) of time points. Moreover, the volume constraint (10) limits the design volume \( V(\phi) \) to a maximum \( V_{\text{max}} \).

Based on the shape sensitivity analysis in [1], a descent direction for the normal velocity \( V_n \) of the evolving boundaries can be formulated with the strain energy density and a Lagrange multiplier \( \lambda \), which corresponds to the volume constraint (10). As indicated in the formulation (8), void elements are filled by a weak ersatz material, see also [1], so that the strain energy density is defined for each element of the discretized elastic body. Hence, the value of the naturally extended normal velocity \( V_n \) on each element \( i \) can be stated as

\[
V_{n,i} = \frac{1}{\mu} d_i - \lambda, \quad \text{with} \quad d_i = \sum_{j=1}^{n} x_i(\phi) u_i^j \mathbf{K}_{i,j} \mathbf{u}_i^j \quad \text{and} \quad \mu = \text{median}(d_1, d_2, \ldots, d_l)
\]

Thereby, the median \( \mu \) is used to make the normal velocity insensitive to the absolute values obtained from the finite element analysis. In the optimization procedure, the normal velocity value of each node can then be calculated, for instance, by averaging the extended normal velocity values of its surrounding elements. Furthermore, the Lagrange multiplier is updated similar to the augmented updating scheme in [15]. For the first optimization time point \( t_1 \), \( \lambda \) \((t_1)\) can be determined by setting a limit for the number of nodes, which receive a negative normal velocity. In further steps, the Lagrange multiplier is calculated as

\[
\lambda(t_{i+1}) = \lambda(t_i) + \gamma(t_i) \Delta \gamma \gamma(t_i), \quad \text{with} \quad \gamma(t_i) = \min(\gamma(t_i-1) + \Delta \gamma, \gamma_{\text{min}})
\]

Thereby, \( \gamma \) is a process parameter, and its initial value \( \gamma(t_1) \), the increment \( \Delta \gamma \), and the upper limit \( \gamma_{\text{max}} \) must be suitably chosen. In this work, these parameters are set as \( \gamma(t_1) = 0.1 \), \( \Delta \gamma = 0.1 \) and \( \gamma_{\text{max}} = 5 \). In case that the velocity field with a selected \( \lambda(t_i) \) does not lead to an appropriate update of the level set function \( \phi \) and the objective function \( \phi \) increases considerably, \( \lambda(t_i) \) can be adapted by controlling the amount of nodes with a negative normal velocity as in the first step. This control method enhances the convergence behavior.

It is worth mentioning that, apart from the steepest descent gradient method, also other approaches can be applied to find the normal velocity. For instance, based on the parametrization (8) and the weakly coupled formulation (9), the gradient calculation as in SIMP-parametrized static problems, see [2], can be considered to determine an alternative normal velocity \( \tilde{V}_n \) as

\[
\tilde{V}_{n,i} = \frac{1}{\mu} d_i - \lambda, \quad \text{with} \quad d_i = \sum_{j=1}^{n} \rho x_i(\phi) u_i^j \mathbf{K}_{i,j} \mathbf{u}_i^j \quad \text{and} \quad \mu = \text{median}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_l).
\]
5. Application Example

The presented optimization method is applied to the well-known example of a planar flexible slider-crank mechanism, see Figure 4. The mechanism consists of a rigid crank with an effective length of 10 cm and a flexible rod with dimensions (40×4×1) and a density of ρ = 8750 kg/m³. The flexible rod is discretized by 20×20 planar bilinear elements. It is worth mentioning that the stress concentration around the joints may lead to high normal velocity values in these areas, which can distort the results of the optimization updating scheme. For this reason, the interface elements of the flexible rod are assumed as rigid, see Figure 4, and its design domain consists of 198×20 inner elements with a reference Young’s modulus of Es = 50 GPa and a Poisson’s ratio of ν = 0.3.

In the transient analysis, the system motion of the flexible slider-crank mechanism is considered within a time period of T = 2 s. Thereby, the rigid crank starts at t = 0 s from the rest position with the angle ϕ(0) = 0 rad, the angular velocity ϕ(0) = 0 rad/s, and is continuously accelerated with a prescribed acceleration ϕ(t) until it reaches the angular position ϕ(2) = 12π rad and the angular velocity ϕ(2) = 12π rad/s at t1 = 2 s. In Figure 5, ϕ(t), ϕ(t) and ϕ(t) are shown. This time simulation is performed in each iteration of the optimization routine and the equivalent static loads at 100 uniformly distributed time points from t = 0 s to t1 = 2 s are then evaluated and used to calculate the objective function J and the normal velocity Vn.

In the optimization routine, the design domain is initially completely filled. The optimization is performed once with the normal velocity Vn (variant 1), see equation (11), and a second time with the alternative normal velocity Vn (variant 2), see equation (13). In Figure 6, the compliance of both variants over the first 100 iteration steps is shown. In both cases, an appropriate convergence behavior can be recognized. The peaks in the compliance history correspond to the iterations, where the Lagrange multiplier λ is adapted by the introduced control method. This improves the convergence behavior. After 100 iterations, the optimization with Vn results in a compliance of J = 0.066 Nmm with a volume ratio of V1 = 0.50, and the optimization with Vn results in a compliance of J = 0.067 Nmm with a volume ratio of V2 = 0.50, respectively. In Figure 7, both optimized variants are shown, which describe two local optiums of a similar quality.

The calculations are all performed on a personal computer equipped with a 3.80 GHz Intel® Xeon® E3−1270 v6 processor and 64.0 GB RAM. The first optimization variant with 100 iterations takes 54.71 min and the second variant 61.73 min. The computational effort is mainly due to the time simulation with an average computation time of 25 s and the preparation of the normal velocity with an average computation time of 7 s in each iteration. Furthermore, the difference between the computation times of both variants is due to the number of required adaptations for the Lagrange multiplier λ, which is equal to 9 in the first optimization variant and 19 in the second optimization variant.
6. Conclusions

The current work shows an efficient procedure for the topology optimization of flexible components in nonlinear dynamic multibody systems by means a radial basis functions based level set method. Thereby, the floating frame of reference approach with global shape functions is used to model the flexible components. The dynamic response, which results from the time simulation, is replaced by equivalent static loads. These are then used to compose the normal velocity of the updating scheme. Using a SIMP method, the elements of the discretized design domain are parametrized, which modifies the equivalent static loads and contributes in this way to the robustness of the optimization. Furthermore, a control sequence is included to avoid instability due to an inaccurately selected normal velocity. In this optimization procedure, two variants of the normal velocity are utilized. As an application example, both variants are then applied to optimize the flexible rod of a slider-crank mechanism. In both cases, an appropriate convergence behavior can be observed and the results possess a similar quality.

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NEW GEOMETRIC FEATURES IN THE TOPOLOGY OPTIMIZATION
FOR ADAPTATION OF STRUCTURES

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Abstract

Adaptation of a structure is done when the body of series products is used commonly in a product family with models having a wide range of mass and stiffness requirements. Also, a series vehicle is often the basis for the development of a competition vehicle, its stiffness and rigidity behavior has to be adapted to this specific use. To do so, reinforcements are added using derivate specific struts and framework structures.

A first method using a SIMP based topology optimization has been presented [1] to identify the locations and layout of the reinforcements and investigations regarding the influence of control parameters have been conducted. Coming from the production costs, one goal for the adaption is, that the connections surfaces between the basis structure and the added structure is small. Therefore we work with a penalization of the sensitivities of structural elements in these positions (boundary zone elements) and we can manage the distance between the basis structure and the added structure.

So far, the method has been developed considering that the loads applied on the basis and the reinforced structures were the same. In order to extend the method also for slightly different and new loads, the control of the geometric features of the reinforcing structure has been developed. These include the geometry of the connection zones, their size and the distance between them. Also the method allows the removal of material that is not forming a member so that the reinforcing structures is only composed of beams.

With these new features, the method also becomes applicable on basis structures that are not issued form a previous optimization but of random topology. Thus, optimized topologies can be created for any kind of adaptation task where the reinforcing structure should be made out of simple beams.

Key words: Topology Optimization, Adaptation, Reinforcement, Geometric Feature, Framework Structures, Vehicle Body

1. Introduction

Automobile companies offer a large range of models to meet the diverse customer needs. In order to reduce production costs, common parts are used whenever possible. For a product family, the vehicle body styles differ especially in the middle, rear and top area regarding the doors, pillars, roofline and cargo area. In contrast, the front and bottom parts remain basically the same within a model range. Examples of configurations include coupe, convertible and station wagon. Each variant has a different basis stiffness and a different final stiffness requirement depending on the motorization and intended vehicle dynamics properties. In order to be able to use the same basis parts, an adjustment of the stiffness can be done by adding reinforcing structures. Under consideration of aspects of the production process like the manufacturing of these reinforcing structures, the ease of assembly and the reparation, framework structure are the preferred construction type to carry out the reinforcements.

In the field of structural optimization, few investigations have been done so far to find the optimal layout of a reinforcing structure. A first publication regarding the optimal stiffener distribution can be found in [2], but no control over features like minimum dimension of the reinforcing structure is considered. Further investigations have been made in [3]. For these optimizations, the design space is restricted to a second overlapping shell and also here no control over the characteristics of the added elements is implemented. These results are useful in order to give the designer hints about where the basis structure should be stiffer in the next model generation. First attempt to enforce
the emergence of beams has also been made using the control options of commercial software.

An overview of adaptation cases depending on the type of basis structure and applied load case is given in Figure 1. A first distinction is made between whether the basis structure is the strict result of an optimization or of random topology. In practice, the topology of the finally produced component is not the optimal structure regarding a single discipline but is a compromise that takes into account further fields. On the other hand a basis structure whose starting performance is too far from the required one would need an amount of adaptation measures that make the common part approach counterproductive. Practical basis structure is located between these two extreme cases.

The adaptation cases can be separated into three categories depending on the applied load case; first, the loads and support might be exactly the same as for the generation of the basis structure, but with higher performance requirements. The second type is when the supports are the same but the load is slightly different. The difference might be regarding the load direction or the application point. Finally a completely new load might be applied. Note that there isn’t a clear separation point between a slightly different and a new load but there is a transition from the one to the other when for example shifting the load application point.

In the case that the basis is random, the first adaptation case per definition does not exist. Still an analogy with an optimized basis can be done for the second and third case. When the applied load is close to the existing load paths of the random basis, the type of loading resembles a slightly different load for an optimized basis. The same analogy applies for the third case where a load far from the existing load paths is applied.

![Types of basis structure and adaptation problem](image)

Figure 1. Types of basis structure and adaptation problem

The challenge with the adaptation task is that the basis structure lies already over some or all the load paths of the new loading problem. Therefore, an optimal reinforcement would be a thickening of some or most of the existing members, depending on the adaptation problem. The addition of a separate framework structure is preferred because of several aspects like costs, manufacturability and assembling.

A method to determine this reinforcing structure under consideration of its geometrical feature is the topic of this investigation.

2. Approach to Solve the Adaptation Problem

The developed method uses the density method Solid Isotropic Material with Penalization (SIMP) [4] for the implementation of the required geometrical features. In order to explain the solving approach, the results obtained with the unmodified SIMP code [5] and with the developed method are shown in Figure 2. The type of basis structure is resulting from a first optimization and the adaptation problem is the same load case with higher
requirement. The basis is generated with a constraint on the volume fraction of 20%, and 30% is defined for the adaptation. The difference between them lies in the distribution of the reinforcing material. The first aspect is the amount of connections between the basis and added material. While with the unmodified code, material is spread over the whole boundary of the basis, only a few connection zones are generated in the new method. Furthermore, a clear distinction is realized in the new result between the connection zones and the distant material forming the reinforcing members. For the connection zones, some geometrical properties are prescribed, which are referred to as features. The height of a connection zone also corresponds to the minimum distance between the basis and reinforcing members. The method to enforce this feature is presented in the first subchapter. Methods to define the size and geometry of the connection zones are the subject of the second subchapter. The third subchapter is about the definition of a minimum distance between the connections. Finally the option to remove benches of material that do not form members is the topic of the fourth subchapter.

![Figure 2. Comparison of an adaptation with unmodified SIMP (left) and the developed method (right)](image)

2.1 Minimum Distance Between the Basis and Distant Members

The approach to enforce the added material to be at a minimum distance from the basis, is to apply a higher penalty exponent for the elements on the boundary of the basis than for the distant elements. The thickness of the boundary zone corresponds then to the desired minimum distance. This way, the ratio stiffness to mass of the boundary zone elements is lower (more mass is required for the same stiffness), which steers the optimizer to use elements from the distant design space to create the reinforcing members. A junction is still necessary between the basis and distant elements so it will be done with as few elements as necessary and at the best locations. A parameter study revealed that the combination of a penalty exponent of 2 for the distant elements and 3 for the boundary zone leads to meaningful results independently of the application case. For the extraction of the boundary zone, a scan procedure has been defined that saves all the elements around the basis within the desired radius. By doing so a minimum value has to be respected to avoid a direct connection through common corner nodes between the basis and distant elements.

2.2 Size and Geometry of the Connections Between the Basis and Reinforcing Structure

One important component of the generated results is the connection zones. In the adaptation case of a slightly different load, the new load paths are close to those of the basis, but shifted. It has been observed that the sensitivities of the elements on the boundary become so high that a higher penalization is not sufficient to generate small connections. One solution to avoid large connection would be to use a maximum length scale filter. Another solution, which is used here, is to take advantage of the fact that the connection areas can be extracted during the optimization iterations and computations can be executed for the elements they contain. First some iteration need to be executed before the connection zones appear and aggregate. Afterward, a search routine has been implemented which finds within the boundary zone all the contiguous elements that fulfill a user defined minimum density and collects them based on neighborhood criteria into different groups. The weighted center of gravity of each group is computed and an orientation for each group can be defined as the direction that points towards the closest element of the basis (see Figure 3). With this information, the size and geometry of a connection can be prescribed. The use of a specific filter for the boundary zone ensures that the minimum thickness of a connection is respected. Any geometry can be enforced by penalizing the elements outside its borders. The simplest one is a circle that is centered at the center of gravity of the group and with its radius as penalization border. Another geometry that has been implemented is a rectangular one, which is defined using the direction of each group towards the basis for its orientation.
2.3 Number of Connections and Distance Between Them

Two processes have been implemented to manage the number of connections and the minimum distance that should separate them, which are the splitting and merging of connection groups. Splitting of groups is required when a connection is large enough to be replaced by several smaller groups under consideration of the geometrical constraints and distance between them. To do so, the width of each connection is computed along with the maximum number of groups that fit in this area. The location of their respective centers of gravity is calculated along with the direction to the basis for each newly generated group. The same process as described in the previous chapter is then applied to generate the correct geometries for the connections.

Merging of connections is required when the distance between the two groups is less than allowed. To identify the groups for which this is the case, the distance between all group centers is checked. In this step, some exception cases have to be considered, for example groups that are located on either side of a basis member must not be merged, even if they fulfill the criteria. Also, two groups that are on each side of a corner should not be merged, even if they are closer to each other than allowed. In order to detect the exception case, an additional computation of the angles between the directions to the basis of proximal groups is executed. Small angles indicate that the groups are on the same side of a straight member. Angles close to 90 degrees are found for groups on each side of a corner and angles around 180 degrees are present for groups on different sides of a member.

2.4 Removal of Local Reinforcement

As observed in some results, reinforcements in the form of localized benches of material might appear despite the use of the so far defined features. It might be beneficial to remove these benches from a cost point of view in trade of a slight loss in stiffness. In order to allow beams only in the final structure, the following process has been defined. The difference between a member and a local reinforcement is that the member starts and ends at either a support or a connection group. By definition a local reinforcement is on the boundary of the basis so it is recognized as a group. Therefore it is possible to define a process by elimination that sorts out the groups that are linked to at least one other or to a support. The remaining groups are those that will be gradually penalized during the last iterations and the abstracted material is spread elsewhere by the optimizer.

3. Examples

The results for the adaptation case “higher requirements” have been presented in Figure 2. For the other adaptation cases, the results of the optimization are presented in Figure 4. It can be observed that for all the cases, the reinforcing structures fulfill the requirements to consist of members attached to the basis at few locations. The slightly different load on a random basis is the most challenging adaptation case. The implemented features act as required but a control over the maximum length scale of the distant member is still required.

The features have been implemented for the computation of 3D models, also based on an MATLAB code [6]. The cantilever problem has been used for the optimized basis and the adaptation case of a higher requirement was applied (see Figure 5). The load case is on the top left, the top middle picture shows the basis structure and the top right the reinforcing one. The bottom picture presents the assembly. The connections between the basis and added structure are also done at few locations and no local reinforcements are present in the final result.

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4. Conclusion and Outlook

The motivation for the adaptation of structures has been explained and an overview of the diverse adaptation cases depending on the type of basis structure and the applied load case has been presented. Several features to control the outcome of the optimization have been successfully implemented and tested on models in 2D and 3D. A further feature that has to be developed is the maximum member size of the distant elements forming the members. The models have been generated using MATLAB codes. In order to be able to use the method on basis structure from external software, the same processes can be applied to the data generated by this software or directly implemented in them.
References


DESIGN OPTIMIZATION OF COMPLIANT MECHANISM CONSIDERING STRAIN RESPONSES

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Abstract

This work deals with the design optimization of compliant mechanisms. We establish a multi-objective topology optimization formulation that considers the output displacement and the strain distribution simultaneously. The topology optimization problem is formulated using the Solid Isotropic Material with Penalization (SIMP) method. The variance of the effective strain quantifies the strain distribution. Since nearly void regions may be highly strained, these regions are excluded in the objective function by a projection method. Svanberg’s MMA method solves the resulting optimization problem, and the required gradients are computed using the adjoint method. We present optimized compliant mechanisms based on linear analysis. Moreover, Pareto optimal solutions have been computed to compare the tradeoff between the end displacement and the structural strain response.

Key words: Topology Optimization, Effective Strain, Compliant Mechanism, Multi-Objective Optimization

1. Introduction

Compliant mechanisms are a class of mechanical systems that replaces conventional hinges in hinge mechanisms by compliant members [1]. Several methods have been used to design compliant mechanisms, including rigid-body replacement method, building blocks method, and topology optimization method. Particularly, topology optimization has become particularly popular because of its ability to capture the elastic behavior of complex geometries [2].

In the design of compliant mechanisms, various objective functions and constraints have been considered: Saxena et al. [3] used the Mutual Potential Energy (MPE) and the Strain Energy (SE) as objectives; Sigmund [4] maximized the Mechanical Advantage (MA), and others include geometrical advantages (GA) [5], characteristic stiffness (CS) [6].

Despite that a large number of methods that have been proposed for mechanism design, hinges do still exist in the optimized mechanisms. The hinges yield large stress concentrations in the structure and thus reduce fatigue life. Many methods have been proposed to eliminate hinges and thereby avoid stress concentrations. Examples of such methods include the minimum length method [7], constraining local, relative rotation [8], and constraining stress/strain [9-10], etc.

There is still a need to develop an effective way to design distributed compliant mechanisms. An essential difference between distributed compliant mechanisms and lumped compliant mechanisms is that, in reality, the deformation of a distributed compliant mechanism is more uniform [6]. Based on this observation, this paper proposes an objective function that describes the effective strain distribution throughout the mechanism. To design compliant mechanisms, we minimize combination of the strain distribution function and negative output displacement. Then, to compare the performance of this new objective, the global effective strain [10] and the maximum effective strain [9] are also utilized as objectives.

2. Three Functions of Describing the Structural Strain Response

This section presents three different functions describing the structural strain response. The effective strain $\varepsilon$ is
computed as
\[
\varepsilon = \frac{1}{1+\mu} \left( \frac{3}{2} \frac{\varepsilon_{\text{dev}}}{\varepsilon_{\text{dev}}^0} \right)^2
\]
(1)
where \( \varepsilon_{\text{dev}} \) is the deviatoric strain, \( \mu \) is the material Poisson’s ratio. It should be noted that the effective strain of void regions should be omitted for three strain response functions mentioned below.

To measure the strain distribution within a structure, let us first consider the strain distribution function \( f_1(\varepsilon) \), which is based on the concept of variance. More precisely, this objective measures the degree of strain dispersion in the design domain. The expression is given as
\[
f_1(\varepsilon) = \frac{1}{|\Omega|} \int_{\Omega} \left( \frac{\varepsilon}{\varepsilon_{\text{me}} - (1 - \gamma)\varepsilon_{\text{me}}} \right)^2 \mathrm{d}V - 1
\]
(2)
where \( \Omega \) represents the design domain, \( |\Omega| \) is the volume of the design domain, \( \varepsilon \) is the scaled effective strain, and \( \varepsilon_{\text{me}} \) is the mean value of \( \varepsilon \). For the strain distribution function, the effective strain in the void region assumes the mean value of effective strain in solid material area. That is, the scaled effective strain is
\[
\varepsilon = \varepsilon_{\text{me}} + (1 - \gamma)\varepsilon_{\text{me}}
\]
(3)
where the \( \varepsilon_{\text{me}} = \frac{1}{|\Omega|} \int_{\Omega} \varepsilon \mathrm{d}V \), \( |\Omega| = \int \mathrm{d}V \) is the volume of the solid regions, \( \gamma \) is the threshold parameter based on the tanh function [11]. It is easily verified that \( \varepsilon_{\text{me}} = \varepsilon_{\text{me}} \).

The second objective that we consider is based on the quadratic sum of the effective strain in the design domain, which is called the global effective strain, and is defined as
\[
f_2 = (\varepsilon) = \int_{\Omega} (\varepsilon)^2 \mathrm{d}V
\]
(4)
where the effective strain \( \varepsilon \) is evaluated from Eq. (1).

The last objective that we consider is the global \( p \)-norm of the effective strain which is used to approximate the maximum effective strain throughout the design domain, and it is defined as
\[
f_3(\varepsilon) = \varepsilon_{\text{PN}} = \epsilon \left( \int_{\Omega} (\varepsilon)^p \mathrm{d}V \right)^{\frac{1}{p}}
\]
(5)
where \( \epsilon \) and \( p \) are parameters. When \( p \to \infty \), then \( \varepsilon_{\text{PN}} \) approaches the maximum effective strain throughout the domain. The normalization parameter \( \epsilon \) is adjusted each iteration to make \( f_3(\varepsilon) \) better approximate the maximum effective strain. The update strategy for \( \epsilon \) is detailed in the work of Le et al. [12]. The role of \( \gamma \) in Eq. (4) and Eq. (5) is to ensure that the effective strain in the void region is eliminated.

### 3. Multi-objective Optimization and Optimization Formulation

The boundary value problem described by Leon et al. [9] is used in this paper. The objective of the optimization is to maximize the output displacement \( u_{\text{out}} \) while minimizing a strain response function \( f \). Here, we will use the response function defined in Eqs. (2), (4), and (5).

The weighted sum method [13] is used to combine two objective functions, and the objective function is stated as
\[
g = -wu_{\text{PN}} + (1 - w)f_{\text{PN}}
\]
(6)
where \( w \) is the weight that reflects the relative importance of the objectives. The normalized strain response function and the output displacement, \( f_{\text{PN}} \) and \( u_{\text{PN}} \) introduced in Eq. (6) are defined as
\[
f_{\text{PN}} = \frac{f_{\text{PN}}}{f_{\text{PN}}} \quad u_{\text{PN}} = \frac{u_{\text{PN}} - u_{\text{min}}}{u_{\text{max}} - u_{\text{min}}}
\]
(7)
where \( f_{\text{PN}} = \max(f(\mathbf{p}^i), f(\mathbf{p}^j)) \) and \( f_{\text{PN}} = \min(f(\mathbf{p}^i), f(\mathbf{p}^j)) \) and \( i \) and \( j \) are obtained by solving the problem (8) with \( g \) replaced by min \( f \) and max \( u_{\text{out}} \), respectively. \( u_{\text{out}} \) and \( u_{\text{out}} \) are obtained in analogy with \( f_{\text{PN}} \) and \( f_{\text{PN}} \). Based on the spring model and finite element analysis, the general formulation is given as
Minimize: \( g \) (a)
Subject to \( K(p)u = f \) (b)
\[ f_i(p) = p^\gamma V - V^* \leq 0 \] (c)
\[ 0 \leq p \leq 1 \] (d)

where \( K \) is the global stiffness matrix, \( f \) is the external load vector, \( u \) is the displacement vector, and \( p \) is the vector of design variables. \( N \) is the number of design variables, \( V^* \) is the allowed volume, and \( V \) is a vector that contains the element volumes. The well-known SIMP model describes the relation between density and material stiffness, i.e., the Young’s modulus of the element \( e \) is obtained as
\[
E_e = E_{\min} + \rho^4 (E_{\max} - E_{\min})
\]
where \( E_e \) is Young’s modulus of the solid phase, \( E_{\min} \) is a small stiffness associated with the void region, and \( k_4 \) is the penalization parameter. Based on Eq. (9), the element stiffness matrix can be calculated as \( K_e = E_e \mathbf{k} \), where \( \mathbf{k} \) is the element stiffness matrix with Young’s modulus equal to 1, and the overall stiffness matrix, \( K \), is assembled through the stiffness matrixes of all elements.

Furthermore, a density filtering technique [14] and a simple projection technique [11] are used in this paper to reduce the mesh-dependence, formation of checkerboard patterns, and gray transition elements problems. The sensitivities are derived using the adjoint method.

4. Numerical Examples

The benchmark tests for compliant grippers cf. Ref. [3] are used to investigate if the presented objectives, \( f_1 \), \( f_2 \), and \( f_3 \), converge towards a design which is in a uniform strain state. We use the design domain shown in Figure 1, where the two dark gray regions represent the non-design domains, with fixed densities. Due to the symmetry of the design domain, only the top half of the mechanism is analyzed. It is discretized into 11,632 four-node rectangular elements with two degrees of freedom at each node. The non-design domains are omitted in the calculation of objective functions (2), (4), and (5). Due to the boundary conditions the non-design domains inherently exhibit large strains, which will inevitably overshadow the strain distribution in all other regions. The geometrical and material parameters, and the optimization parameters are listed in Table 1.

![Figure 1. Design domain of the compliant gripper problem [2]](image)

Table 1. Parameters of the Compliant Mechanism (dimensionless parameters)

<table>
<thead>
<tr>
<th>Property</th>
<th>Quantity</th>
<th>Iteration parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ( L )</td>
<td>160</td>
<td>Maximum volume fraction ( V^* )</td>
<td>0.3</td>
</tr>
<tr>
<td>Poisson ratio ( v )</td>
<td>0.3</td>
<td>Filter radius ( R )</td>
<td>6 times of element length</td>
</tr>
<tr>
<td>External load ( F_e )</td>
<td>1</td>
<td>Penalty coefficients ( k_4 )</td>
<td>increased from 1 to 5 for ( f_1 ) and ( f_2 ), increased from 1 to 30 for ( f_3 )</td>
</tr>
<tr>
<td>Output spring ( k_{out} )</td>
<td>0.1</td>
<td>Projection method for strain response ( \eta ) and ( \beta )</td>
<td>10 iterations until 100 from 1</td>
</tr>
<tr>
<td>Input spring ( k_i )</td>
<td>0.1</td>
<td>Projection filtering technique ( \eta ) and ( \beta )</td>
<td>( \eta = 0.5 ), ( \beta ) is doubled every 30 iterations until 512 from 1</td>
</tr>
<tr>
<td>Young’s modulus ( E_e ) (solid) and ( E_{\min} ) (void)</td>
<td>1 and ( 10^{-9} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 2 shows Pareto optimal solutions obtained from solving the problem with strain response function \( f_1 \), \( f_2 \), and \( f_3 \), respectively. The competitive relationship between the two objectives can be seen clearly, the output displacement decreases when the strain responses value decreases, which proves that the multi-objective optimization
method is effective.

(a) Strain Distribution Function  (b) Global Effective Strain  (c) Maximum Effective Strain

Figure 2. Pareto optimal solutions of the compliant gripper problem obtained by using different function

(a) $u_{opt}=1.10, f_1=0.680, f_2=0.83, f_3=0.083, \xi=0, 30, k=6.1$  
(b) $u_{opt}=0.84, f_1=0.031, f_2=0.76, f_3=0.069, \xi=0, 24, k=4.5$

(c) $u_{opt}=1.02, f_1=0.080, f_2=0.59, f_3=0.069, \xi=0, 30, k=6.1$  
(d) $u_{opt}=0.98, f_1=0.112, f_2=0.54, f_3=0.041, \xi=0, 30, k=4.0$

Figure 3. Optimized topologies and strain distributions obtained by using different objective functions under the assumption of small strain linear elasticity. Lumped hinges appear when using $-u_{out}$ as the objective function as shown in Figure 3(a). Comparing Figure 3(a) and Figure 3(b), we see that using multi-objective instead of only considering the end displacement $-u_{out}$ results in a much more even strain distribution. The lumped hinges are eliminated, and the structure becomes thinner and uniform across the cross-sectional area. Figure 3(c) shows the topology obtained by using the global effective strain $f_2$ in Eq. (4), one can find that the new lumped hinges emerge in the central domain even though $f_2$ is getting smaller. Figure 3(d) shows the topology obtained by using the maximum effective strain $f_3$ in Eq. (5), where the lumped hinges are eliminated and a distributed compliant mechanism is generated, but unlike the topology in Figure 3(b), the structure is not always thin and the effective strain is not uniformly distributed throughout the whole structure. A muscle-like structure is designed.

5. Conclusions

This study concludes that the choice of the strain distribution function is closely connected to the distribution of the structural strain. Moreover, by carefully choosing the objective, the lumped hinges can be avoided and a robust compliant mechanism can be obtained. In the results obtained by using the global effective strain and the maximum effective strain, we find that the global effective strain method may render lumped hinges. However, the topology obtained by using the maximum strain method has poor strain distribution which is manifested by a large value of the strain distribution function.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (NSFC, Nos. 51505064,
11711530018). Fundamental Research Funds for the Central Universities (DUT17ZD207), the Swedish Foundation for International Cooperation in Research and Higher Education (STINT, No. IB2018-7470), and the Swedish strategic research program (eSSENCE). These supports are gratefully appreciated.

References

CONCURRENT DESIGN OF ADDITIVE MANUFACTURING-ORIENTED COATED STRUCTURES WITH GRADED LATTICE INFILL THROUGH EXPLICIT TOPOLOGY OPTIMIZATION

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Abstract

In the present work, a new approach for designing additive manufacturing-oriented coated structures with graded lattice infill simultaneously is developed based on the Moving Morphable Components/Voids (MMC/MMV) topology optimization framework. To this end, a set of morphable voids is adopted to describe the boundary of the exterior coated solid shell, while a set of morphable components combing with a coordinate perturbation technique is introduced to represent the graded infill material distribution. With the use of the proposed approach, both the coated solid shell and the layout of the graded infill can be optimized simultaneously with a very small number of design variables. Numerical examples demonstrate the effectiveness of the proposed approach.

Key words: Topology Optimization, Additive Manufacturing, Coated Structure, Graded Infill, Moving Morphable Component/Void (MMC/MMV)

1. Introduction

Lattice/porous structures with well-designed periodic microstructures possess excellent mechanical, thermal, optical and acoustic properties. Nowadays, with the rapid development of modern manufacturing technology (e.g., additive manufacturing), it is no longer difficult to fabricate lattice/porous structures with complicated microstructures. Furthermore, many theoretical and experimental studies have shown that lattice/porous structures constituted by graded microstructures have many advantages compared with the ones composed of uniformly periodic microstructures. Actually, this point can be seen very clearly by investigating the internal structures of many natural materials. One of the promising examples is that the internal fibers of the bamboo are ingeniously distributed in a gradient way to strengthen the bending stiffness very effectively. Therefore, recent years witnessed a growing interest in developing systematic and efficient methods to design lattice/porous structures with graded microstructures.

Several methods have been proposed to design graded lattice/porous structures[1-3]. However, in most of the previous works, the obtained graded microstructures are usually in the form of self-similar unit cells with discontinuous values of geometry parameters along a prescribed spatial direction. Obviously, besides the substantial reduction of the design space, this treatment will inevitably lead to a configuration mismatch between two adjacent cells, and tedious post-processing is often required to render the smoothness of the optimized designs for real world applications[3].

Another limitation in the works mentioned above is that the manufacturing requirements have not been fully taken into consideration in the corresponding problem formulations. For example, in order to balance the fabrication cost and mechanical properties of the manufactured structures, in industrial 3D printing process (e.g., selective laser melting), the printed structures are usually designed in a Shell-Infill form, where a solid shell constituting the exterior part of the structure (coating surface) while a lattice infill distributed in the structural interior. Under this circumstance, except for the infill microstructure (possibly distributed in a graded pattern), the coating shell should also be considered during the design stage. In order to resolve this issue, more general approaches that can be used to optimize the configuration of the coating shell as well as the layout of the spatially graded infill structures simultaneously have been developed recently [4-5]. Unfortunately, the upgraded methods also suffer from the issues in their original counterparts such as difficulty for controlling of graded infills and mismatch between adjacent unit cells.
In the present work, a new approach is developed for designing coated structures with graded lattice infill, where a solid exterior shell and the corresponding graded lattice infill, can be optimized simultaneously. In order to achieve that, the MMC [6] and MMV [7] based methods are combined together for developing the optimization problem formulation and the solution algorithm. To be specific, in the proposed approach, moving morphable voids described by B-splines are employed to represent the solid coating shell, while the moving morphable components described by p-th order superellipses are used to construct the lattice/porous infills. As will be demonstrated in the forthcoming sections, with the help of coordinate perturbation technique, both the configurations of the solid shell (coating) and complex graded infills can be optimized simultaneously using a very small number of design variables in the proposed approach. Numerical example demonstrates the effectiveness and efficiency of the proposed approach.

2. Geometrical Model of a Structure with Solid Coating Shell And Graded Lattice Infill

2.1 Explicit Geometry Description of a Graded Lattice Structure

In this subsection, the TDF characterizing the geometry of a graded lattice structure is constructed based on the MMC method. The basic idea is to introduce the so-called coordinate perturbation functions (CPF$\phi$) to the TDF of a prototype periodic lattice structure, which is based on the observation that graded structures can be formed through a coordinate transformation from a periodic structure.

![Diagram of Coordinate Transformation](image)

Figure 1. Generating a graded lattice structure through coordinate transformation from a periodic structure.

The prototype lattice structure can be generated through duplicating a primitive unit cell periodically in the prescribed design domain $D$. The material distribution in the unit cell can be represented by a number of components distributed in it, for example, $m$ components in the primitive unit cell. Assuming that the coordinates of the central point of the $k$-th component in the primitive generation cell are $(x_{k,i}, y_{k,i})$, as shown in Figure 1a, the coordinates of the central point at the corresponding component at cell $(i,j)$ are $(x_{k,i}, y_{k,i})$ with $x_{k,i} = x_{i,j} + (i - 1)x_0$, $y_{k,i} = y_{i,j} + (j - 1)y_0$, $x_0$ and $y_0$ denoting the dimensions of the generation cell along two coordinate directions, respectively. Then the TDF of the corresponding component at cell $(i,j)$ can be expressed as

$$
\phi_{k,i}(x, y) = 1 - \left( \frac{x}{a} \right)^{\alpha} + \left( \frac{y}{b} \right)^{\beta}
$$

(1)

with

$$
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_x & \sin \theta_y & x - x_{i,j} \\
-\sin \theta_x & \cos \theta_y & y - y_{i,j}
\end{bmatrix}
$$

(2)

The TDF of the periodic lattice structure can be constructed as

$$
\phi^{\text{p} \rightarrow \text{i}}(x) = \max_{i=1}^{m} \max_{j=1}^{m} \phi_{k,i}(x)
$$

(3)

The next step is to introduce the coordinate perturbation operation to $\phi^{\text{p} \rightarrow \text{i}}(x)$. Borrowing the concept from continuum mechanics, the corresponding coordinate transformation can be expressed as:

$$
\begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix} = \begin{bmatrix}
1 & f(x, y) \\
0 & g(x, y)
\end{bmatrix}
$$

(4)

where $(x, y)$ and $(\tilde{x}, \tilde{y})$ denote the coordinates of the lattice structure before and after transformation, respectively. The symbols $f(x, y), g(x, y)$ are the coordinate perturbation functions, with
\[
\begin{align*}
\frac{f(x, y)}{g(x, y)} &= \sum_{i=1}^{n} a_i \phi_i(x, y) \\
\frac{g(x, y)}{g(x, y)} &= \sum_{i=1}^{n} \beta_i \phi_i(x, y)
\end{align*}
\]

(5)

where \(a_i, \beta_i\) are some coefficients; \(\phi_i(x, y)\), \(\phi_i(x, y)\) are the basis perturbation functions of \(x\) and \(y\) coordinates. They could be of trigonometric, polynomial, exponential forms, respectively. In the present paper, it is assumed that \(\phi_i(x, y) = \phi_i(x)\), \(\phi_i(x, y) = \phi_i(y)\) for simplicity and only trigonometric base functions are adopted in numerical implementation. For example, one can choose

\[
\begin{align*}
\frac{f(x)}{g(y)} &= \sum_{i=1}^{n} a_i \phi_i(x) + a_z \phi_z(x) \\
\frac{g(y)}{g(y)} &= \sum_{i=1}^{n} \beta_i \phi_i(y) + \beta_z \phi_z(y)
\end{align*}
\]

(6)

with

\[
\begin{align*}
\phi_i(x) &= \cos\left(\frac{(r-1)\pi}{l} \left( x - \frac{l}{2} \right) \right), \quad \phi_z(x) = \sin\left(\frac{(r-1)\pi}{l} \left( x - \frac{l}{2} \right) \right) \\
\phi_i(y) &= \cos\left(\frac{(r-1)\pi}{h} \left( y - \frac{h}{2} \right) \right), \quad \phi_z(y) = \sin\left(\frac{(r-1)\pi}{h} \left( y - \frac{h}{2} \right) \right)
\end{align*}
\]

(7, 1, 2)

where \(l\) and \(h\) are the characteristic length and width of the design domain, respectively. Based on the above discussions, the TDFs of the graded lattice structure (see Figure 1b for reference) can be expressed as

\[
\phi^{x-y}(x, y) = \phi^{x-y}(\tilde{x}, \tilde{y})
\]

(8)

It should be noted that the coordinate system chosen for geometry description can be polar, cylindrical, spherical or even hyperbolic ones. The readers are referred to [8] for more details.

## 2. 2 Explicit Geometry Description of a Shell-Infill Structure

For the graded structures generated in subsection 2.1, the lattice is distributed in the whole design domain. In this sense, the advantage of ‘topology’ optimization, which can introduce holes creatively in the design domain, is not fully enjoyed. To improve this issue, for a base lattice structure with TDF \(\phi^{x-y}\), (as shown in Figure 2a), some morphable voids with thickness can be introduced into the base structure to optimize the filled region. According to this, the boundaries of the morphable voids are then constructed as solid shells with a certain thickness, and the graded structures with solid coating shell and lattice infill can be achieved through the following steps.

Figure 2. A schematic illustration of generating the TDF of a shell-infill graded lattice structure.

The first step is to generate a solid structure by the MMV method, as shown in Figure 2b, and the TDF of this solid structure denoted as \(\phi_0\). Thanks to the explicit description of voids (i.e., \(D = (x_{0j}, y_{0j}, r_j)\)) in the MMV method, it is easy to expand the boundaries of the \(j\)-th void by perturbing the distance vector \(r_j = (d_j', \cdots, d_j')\) to \(r_j = (d_j', \Delta d_j, \cdots, d_j' + \Delta d_j)' , j = 1, \cdots, n_v\). This treatment yields an intermediate structure with TDF \(\phi_0^{\Delta}\) illustrated in Figure 2c.
as well as its dual structure as Figure 2d with TDF $-\phi'$. In the following step, the domain inserted with the infill part shown in Figure 2e can be described by the TDF $\phi' = \max(-\phi, \phi')$. Finally, the TDF of the entire coated structure with graded lattice infill illustrated in Figure 2f can be constructed through the operation $\phi' = \min(\phi, \phi')$. In summary, the TDF of the coated structure with graded lattice infill can be calculated by

$$\phi' = \min(\phi, \max(-\phi, \phi'))$$

(9)

It is also worth noting that the parameter $\Delta d$ corresponding to the thickness of the solid shell is prescribed fixed in the present work. Actually, the value of $\Delta d$ can also be optimized to achieve the so-called coating thickness design with the use of the proposed approach. Compared with the treatment in previous works where several intermediate fields or multiple smoothing and projections [9] are required, the proposed approach can achieve the same goal in a more straightforward way.

3. Problem Formulation

Based on the above discussions, it can be concluded that the vector of the design variables for optimizing the coated structure with graded lattice infill is $D = ((D_{\text{solid}})^{T}, (D_{\text{inf}})^{T})^{T}$ with $D_{\text{solid}} = ((D_{1}^{T}), \ldots, (D_{n}^{T}), (D_{m}^{T})^{T})^{T}$ and $D_{\text{inf}} = ((D_{1}^{T}), \ldots, (D_{n}^{T}), (D_{m}^{T})^{T})^{T}$, respectively. The symbol $D_{i}^{T} = (x_{i}, y_{i}, \alpha_{i}, \beta_{i}, D_{i}^{T})$ includes the parameters describing geometry of the $k$-th component in the generating cell (i.e., cell $(1,1)$) and $d_{i}$ is the vector of the parameters associated with the width function $b_{i}(x)$. The vector $D_{\text{inf}} = ((\alpha)^{T}, (\beta)^{T})^{T}$ with $\alpha = (a_{1}, \ldots, a_{n}, a_{0})^{T}$ and $\beta = (\beta_{1}, \ldots, \beta_{n}, \beta_{0})^{T}$ contains the coefficients associated with the perturbation basis functions along with two coordinate directions. Besides, $D_{i}^{T} = (x_{i}^{0}, y_{i}^{0}, r_{i})^{T}$ with $r_{i} = (d_{i}, \ldots, d_{i})$ represents the geometrical parameters describing the shape of the $j$-th void.

In the present study, the coated structures with graded infill are designed to minimize the structural compliance under the volume constraint of available solid material. Under this circumstance, the corresponding problem formulation can be written as:

Find $D = ((D_{\text{solid}})^{T}, (D_{\text{inf}})^{T})^{T}$, $u(x) \in H^{1}(D)$

Minimize $C = \int_{D} H(\phi'(x;D)) f \cdot u dV + \int_{\Gamma} t \cdot u dS$

subject to

$$\int_{D} E(H(\phi'(x;D))) : g(u) : g(v) dV = \int_{D} H(\phi'(x;D)) f \cdot v dV + \int_{\Gamma} t \cdot v dS, \quad \forall v \in u_{w}$$

$$V_{u} = \int_{D} H(\phi''(x;D)) dV \geq V$$

$$D \subset u_{D}, \quad u = \bar{u}, \quad \text{on} \quad \Gamma_{u}$$

where $D$, $f$, $t$ and $\bar{u}$ are the design domain, body force density, prescribed surface traction on Neumann boundary $\Gamma$, and prescribed displacement on Dirichlet boundary $\Gamma_{u}$, respectively. In Eq. (10), $H^{1}(D)$ denotes the first-order Sobolev space and $\phi'(x;D)$ is the TDF of the whole structure, $\phi''(x;D)$ is the TDF of the intermediate structure of Figure 2c, $V_{u} \geq V$ is the fill volume constraint, which restricts that the volume of the region occupied by the filled structures between the inner and outer boundaries should be no smaller than the given volume $V$. The symbol $H = H(x)$ denotes the Heaviside function with $H = 1$ if $x > 0$, otherwise $H = 0$. Besides, $E = E(H(\phi'(x;D)))$ is the fourth order elasticity tensor of the material locating at point $x$. In Eq. (10), $u_{w} = (v | v \in H^{1}(D), v = 0 \text{ on } \Gamma_{u})$ is the admissible set of the virtual displacement vector $v$ and $\bar{V}$ is the upper limit of the volume of the available solid material.

4. Numerical Examples

In this section, an MBB example is investigated to illustrate the effectiveness of the proposed method. Without loss of generality, all involved quantities are assumed to be dimensionless, and the thickness of the design domain is unit. The Young’s modulus and the Poisson’s ratio of the isotropic solid material are chosen as $E = 1$ and $\nu' = 0.3$, respectively.
The setting of this MBB example is described schematically in Figure 3a. A vertical load is imposed on the middle point of the top side with the magnitude of \( F = 1 \). The thickness of the pre-set boundary shell is \( \Delta d = 0.06 \). The upper bound of the volume of available solid material is set as \( \overline{V} = 0, 3V_b \), meanwhile, four cases with the lower bounds of the fill volume \( \underline{V} \) equalling to 0, 5V_b, 0, 6V_b, 0, 7V_b, 0, 8V_b are tested. The objective is to design an optimized structure for minimizing compliance.

![Figure 3. The MBB beam example.](image)

The same initial design is used for all the four cases. As shown in Figure 3b, the design domain contains 64 (along the width direction) by 16 (along the length direction) lattice cells, with 2 components in the primitive cell. The 12 internal “voids” and the outer boundary of the initial design both are represented by B-spline curves, with thickness \( \Delta d = 0.06 \), which is fixed during the entire optimization process. In addition, a 960 \( \times \) 240 uniform Q4 mesh is used for finite element analysis.

![Figure 4. Optimized results of the MBB beam example with respect to different fill volume constraints.](image)

With the use of the global coordinate perturbation formulated in Eq. (6) to Eq. (7) \( (r=4, l=4) \), the optimized shell-infill graded lattice structures for these four cases are shown in Figure 4a–Figure 4d, respectively. It can be observed clearly that all optimized structures are composed of graded infills varying along with multiple directions. Meanwhile, the connection between adjacent unit cells are also quite smooth and natural due to the use of the coordinate perturbation technique. In addition, it is also found that the as value of \( \overline{V} \) increases (i.e., from 0, 5V_b to 0, 8V_b), the optimized value of structural compliance is also increased (i.e., from 114, 57 to 197, 82). This is consistent with the observations made in Wu et al. [4] such that the lattice-like structures are actually inferior to the structures without porous microstructures when only overall stiffness is considered. Nevertheless, it is worth noting that the coated structures with lattice infills are in general more efficient to preserve structural stability under compressive load, more robust under load/material uncertainty and has much better energy absorption capability. It can be expected the proposed approach can also find its applications when structural stability, load/material uncertainty and other porous microstructure-friendly structural performance measures are considered.

One particular advantage of the present topology optimization method is the optimized designs are explicitly described and can be transferred to CAD system directly to rebuild the structure without any postprocessing. For illustration, Table 1 lists the coordinates of the control points associated with the outer boundary of “void” \( c_i \) shown in Figure 4a. It is also worth noting that the optimized coated structure with graded infill is obtained with a small number of design
variables (actually only $4 \times 2 + 8 \times 2 + 14 \times 13 = 206$). Furthermore, the number of the design variables can be further reduced by taking the symmetry of the problem into account. Generally, such designs cannot be obtained easily through traditional implicit topology optimization methods by using the same order of the number of design variables.

Table 1. Coordinates (relative to the coordinate system shown in Figure 3(a)) of the Control Points for the B-spline Boundary curve Shown in Figure 4(a).

<table>
<thead>
<tr>
<th>Control Point Number</th>
<th>Coordinate</th>
<th>Control Point Number</th>
<th>Coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(5, 3974, 2.3705)</td>
<td>7</td>
<td>(1, 7537, 2.3705)</td>
</tr>
<tr>
<td>2</td>
<td>(4, 7928, 2.6627)</td>
<td>8</td>
<td>(0, 3382, 0.0909)</td>
</tr>
<tr>
<td>3</td>
<td>(4, 5192, 2.7733)</td>
<td>9</td>
<td>(3, 0171, 0.1715)</td>
</tr>
<tr>
<td>4</td>
<td>(4, 2867, 2.7961)</td>
<td>10</td>
<td>(4, 2867, 0.0682)</td>
</tr>
<tr>
<td>5</td>
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<td>11</td>
<td>(5, 5678, 0.1516)</td>
</tr>
<tr>
<td>6</td>
<td>(2, 0472, 3.6635)</td>
<td>12</td>
<td>(8, 3374, 0.0318)</td>
</tr>
</tbody>
</table>

5. Conclusions

In the present work, a new approach for designing coated structures with graded infill is developed with the use of the Moving Morphable Components/Voids (MMC/MMV)-based topology approaches. The key idea is to describe the geometry of the considered structure using a set of geometry parameters. Under such treatment, both the crisp boundary of the exterior coating shell and the configuration of continuously distributed graded infill can be optimized simultaneously, without any post-processing, only using a very small number of design variables. Future research directions deserve intensive explorations including the design of three-dimensional coated structures with graded lattice infill considering manufacturing constraints/uncertainties, design optimization of lattice type metamaterial, and optimal design of lattice structures considering buckling constraint and multi-physics requirements, etc. Corresponding results will be reported in separated works.

Acknowledgments

The financial supports from the National Key Research and Development Plan (2016YFB0201600, 2016YFB0201601), the National Natural Science Foundation (11732004, 11821202), Program for Changjiang Scholars, Innovative Research Team in University (PCSIRT) and 111 Project (B14013) are also gratefully acknowledged.

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HIGH-ORDER SENSITIVITY ANALYSIS FOR TOPOLOGY OPTIMISATION

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Abstract

Almost all topology optimization algorithms use linear sensitivity analysis where the effects of updating design variables on objective functions are approximated linearly. This paper explores the use of higher order sensitivity analysis. Whilst computationally more expensive, higher order sensitivity analysis allows implementing larger changes in design variables which could be beneficial in some applications.

Focusing on displacement-based objective functions, and particularly mean compliance, higher order derivatives of the objective function are formulated based on FEA discretisation. An approach to calculate sensitivities to any desired accuracy is presented. This calculation allows one to accurately predict the responses of the system after arbitrary large changes in design variables.

Some numerical issues involved in calculating higher order sensitivities are discussed. Solutions are proposed to address these numerical issues and to calculate the higher order sensitivities more efficiently and robustly.

Key words: Optimisation, Sensitivity Analysis, High Order Derivatives

1. Introduction

Engineering design problems can be formulated as constrained multi-objective optimisation problems. In structural design, at least one of the objective functions is defined based on mechanical responses of the system, most commonly deformations. To solve the resulted optimisation problem, one needs to have a measure of the sensitivities of this objective function to different changes in the elements of the system. Gradient-based optimisation methods generally only require the gradient of the objective function or first-order sensitivities. However, the gradient only provides a linear approximation of the changes in the objective function and thus is only useful when the changes in the design variables are small. When the changes in design variables are large, higher order sensitivity analysis may be required. Higher order sensitivity information can also be helpful where design variables are not changing continuously (see e.g. [1]).

There is a very limited volume of work on using higher order sensitivities. The two main discouraging factors seem to be the complexities involved in using high order sensitivities and their computational cost. This paper tries to pave the way towards simpler and easier calculation of higher order sensitivities. Admittedly, the computational cost remains a barrier. However, it should be noted that the main benefit obtained by using high order sensitivities is that changes in the system due to an arbitrary large change in variables can be estimated to any desired accuracy.

At first formulae are presented for all the derivatives of a general linear function of displacements in form of Taylor’s (polynomial) series. Then the convergence properties of this series is investigated and it is shown that the polynomial series are not necessarily convergent. To solve this problem, exponential series are used and it is shown by a simple example that they can converge faster than the polynomial series and don’ show the divergence problem observed in polynomial series.
2. Design Variables and Objective Function

Consider a design domain with prescribed loads and supports. For simplicity we limit ourselves to linear problems in this study. Using finite element discretisation, the equilibrium equations of the system takes the following form.

\[ Ku = f \]

with \( K \) denoting the global stiffness matrix of the system, and \( f \) and \( u \) representing the nodal force and displacement vectors respectively. The number of elements is denoted by \( n \).

We assume that the stiffness matrix \( K \) is a linear function of a number of design variables in the following form.

\[ K(x) = K_0 + \sum_{i=1}^{n} x_i K_i \]

Here \( x = [x_1, \cdots, x_N]^T \) is the vector of design variables and \( K_i \) are preselected matrices all having the same size as \( K \), i.e. \( d \times d \) where \( d \) is the total number of degrees-of-freedom in the system.

Note that \( N \) doesn’t necessarily have to be equal to \( n \). However, setting \( N = n \) and selecting \( K_i \) as the (global level) stiffness matrix of element \( i \) results in the commonly employed element-based parametrisation in sizing or topology optimisation. It should also be noted that the design variables should satisfy certain constrains to ensure that the above parametrisations don’t lead to mathematically or physically infeasible stiffness matrices.

The following linear function of displacements is considered as the objective function

\[ f = h^T u \]

Here \( h \) is a \( d \times 1 \) predefined fix vector.

3. Sensitivity Analysis

Using Taylor series, the change in \( f \) due to a change in the value of \( x \), can be expressed as

\[ \Delta f = \frac{\partial f}{\partial x_i} \Delta x_i + \frac{1}{2!} \frac{\partial^2 f}{\partial x_i^2} \Delta x_i^2 + \cdots + \frac{1}{m!} \frac{\partial^m f}{\partial x_i^m} \Delta x_i^m + \cdots \]

Noting that \( h \) is fixed and using the linearity of \( (2) \), following [1] we obtain

\[ \frac{\partial^m f}{\partial x_i^m} = m! h^T \left( -K^{-1} \frac{\partial K}{\partial x_i} \right)^m K^{-1} f \]

Using (2) in the above equation, we have

\[ \frac{\partial^m f}{\partial x_i^m} = m! h^T \left( -K^{-1} K_i \right)^m K^{-1} f \]

(6)

Based on (6), one can calculate any higher order term from the previous one by simple matrix multiplications. The main issue in eq. (6) is calculating and storing the compliance matrix \( K^{-1} \) which can be extremely expensive particularly in large systems. However, if we can calculate the changes in the system due to an arbitrary change and to any desired accuracy, theoretically we can directly update the compliance matrix without reanalysis. This can be of particular importance in non-linear systems.

Although (6) allows us to calculate all the derivatives of the objective function, note that the polynomial series in (4) might converge very slowly or not at all. Focusing on the mean compliance as the objective function in the next section, convergency of this series is studied.

4. Convergence of the Polynomial series

Consider mean compliance as the objective function. Mean compliance is a special case off in (3), obtainable by setting \( h = f \), i.e. \( c = f^T u \).

Assume an element based parametrisation of \( K \) in (2), with \( N = n, K_0 = 0, \) and \( K_i, i = 1, \cdots, n \) matrices as stiffness matrices of elements 1 to \( n \) which are all made of a base isotropic linear elastic material with Young’s modulus of \( E = 1 \) and Poisson’s ratio of zero.

The Zhou-Rozvany problem [2] is considered here with the design domain depicted in Figure 1. This is a small plane
stress problem with only 100 elements. Initially we have \( x_0 = 1 \) in all elements. All the units are consistent.

![Diagram](image)

Figure 1. The Zhou-Rozvany problem. Element numbers are noted inside elements. Moving \( x_{06} \) from 1 to 0 Using Taylor Series Moving \( x_{15} \) from 0 to 1 Using Taylor Series

![Graphs](image)

Figure 2. Estimating the changes in \( \epsilon \) due to changes in \( x_{06} \) using the Taylor series (4). \( m \) is the number of terms used in the series.

Variations of \( \epsilon \) with respect to changes in the design variable associated with element 95 (\( x_{06} \)) are studied first. Two cases are considered; a) \( x_{06} \) decreases from 1 to 0; and b) \( x_{06} \) increases from 0 to 1. The results are shown in Figure 2.

Looking at Figure 2(a), the Taylor series is converging in case (a) and gives a reasonable result after considering 9 terms with a relative error of about 1.01\( \times 10^{-4} \). In fact for any \( -1 \leq \Delta x < 0 \) it can be shown that the series (4) is convergent for mean compliance [1]. For case b, however, Figure 2(b) reveals that the polynomial series is clearly diverging.

5. Using Exponential Series

To solve the convergency issue observed in the previous section, the exponential series can be used instead of polynomial series. To achieve this, the objective function is expressed in terms of a series of exponentials in the following form

\[
 f(x) = \sum_{i=1}^{\infty} a_ie^{i\epsilon}. \tag{7}
\]

Appendix A introduces a simple approach to calculate the coefficients of this series. Using the exponential series the two cases shown above are solved again. The results are shown in Figure 3. From these Figures, it is clear that for both cases the exponential series is convergent. Using 5 terms for \( \Delta x_{06} = 1 \) results in a relative error of 1.54\( \times 10^{-4} \). For \( \Delta x_{06} = -1 \) the relative error of 4.81\( \times 10^{-4} \) is obtained after using 4 terms in (7). It should be noted that in terms of computational cost, an exponential series with m terms is comparable to a Taylor series with 2m terms.
To check the effectiveness of the exponential series in dealing with large changes, elements 33 to 60 are all changed together. Assuming \( x_{33} = x_{34} = \cdots = x_{60} = x \), the value of \( x \) is once reduced from 1 to 0 and then increased from 0 to 1. The results are shown in Figure 4.

Looking at Figure 4(a), when reducing the design variables the series converges rapidly with only 5 terms providing an acceptable relative error of around \( 1.28 \times 10^{-3} \). However, Figure 4(b) shows that when increasing \( x \) from 0, the convergence is not monotonic and using 12 terms results in better approximation than 15 terms. Furthermore, even after using 15 terms, a good approximation cannot be achieved. The problem, however, seems to be in the small initial value of 0. Starting from \( x = 0.1 \) and increasing it to 1, the results are shown in Figure 4(c). In this case 5 exponential terms provide a good approximation with a relative error of about \( 1.93 \times 10^{-3} \).

Moving \( x_{33} \) from 1 to 0 Using Exp Series        Moving \( x_{33} \) from 0 to 1 Using Exp Series

Moving \( x \) from 1 to 0 using exp series        Moving \( x \) from 0 to 1 using exp series        Moving \( x \) from 0.1 to 1 using exp series

Figure 3. Estimating the changes in \( c \) due to changes in \( x_{33} \) using the exponential series (7).

\( m \) is the number of terms used in the series.

Figure 4. Estimating the changes in \( c \) due to changes in \( x_{33} = x_{34} = \cdots = x_{60} = x \) using the exponential series (7).

\( m \) is the number of terms used in the series.
6. Concluding Remarks

1. Formulae are proposed for high order sensitivity analysis of a linear function of displacements. These formulae have an iterative nature but require calculation and storage of the compliance matrix.
2. It was shown that the Taylor series approximation of the objective function is not necessarily convergent.
3. A different approximation based on exponential series is introduced which doesn’t have this convergency problem and generally converges faster than Taylor series.
4. A simple method is proposed to efficiently calculate the coefficients of this exponential series.

References


Approximating a Function Using Exponential Series

Suppose that the value of a function \( f(x) \) and all its derivatives at \( x = 0 \) are known. These are denoted by \( \varphi_j = f^{(j)}(0) \). The following exponential series is assumed to approximate the function at a neighbourhood of \( x = 0 \).

\[
    f(x) = \sum_{i=1}^{n} a_i e^{t_i x}
\]  

(8)

We are interested in finding \( a_i \)s and \( t_i \)s in terms of \( \varphi_j \). From (8), \( \varphi_j = \sum_{i=1}^{n} t_i a_i, j \in \{0, 1, \ldots, m-1\} \) Assuming a truncated version of (7) with \( m \) terms. \( 2m \) equations are needed to solve for \( a_1, \ldots, a_m \) and \( t_1, \ldots, t_m \). This gives us the following system of \( 2m \) nonlinear simultaneous equations.

\[
    \varphi_j = \sum_{i=1}^{n} t_i a_i, \quad j \in \{0, 1, \ldots, 2m - 1\}
\]  

(9)

To solve this system, we first try to find the coefficients of the following polynomial whose roots are the \( t_1, \ldots, t_m \).

\[
    P(t) = (t-t_1)(t-t_2)\cdots(t-t_m) = t^m + p_1 t^{m-1} + p_2 t^{m-2} + \cdots + p_m
\]  

(10)

The coefficients \( p_1, \ldots, p_m \) are related to the roots of (10) through the following relationships

\[
    p_1 = -\sum_{i=1}^{m} t_i
\]

\[
    p_i = \sum_{j=1}^{m} t_j
\]

\[
    \vdots
\]

\[
    p_m = (-1)^m \prod_{i=1}^{m} t_i
\]  

Looking back at (9) and multiplying every equation by \( t_i \) and subtracting the next equation from it, we obtain

\[
    t_i \varphi_{j-1} - \varphi_j = \sum_{k=1}^{m} t_i^{k-1} (t_k - t_i) a_k, \quad j \in \{1, \ldots, 2m - 1\}
\]  

(12)

Now multiplying every equation in (12) by \( t_j \) and subtracting the next equation from it, we obtain

\[
    t_j (t_i \varphi_{j-1} - \varphi_j) - (t_j \varphi_{j-1} - \varphi_j) = t_j t_i \varphi_{j-2} - (t_i + t_j) \varphi_{j-1} + \varphi_j = \sum_{k=1}^{m} t_j^{k-2} (t_k - t_j) t_j a_k, \quad j \in \{2, \ldots, 2m - 1\}
\]  

(13)

Each time this procedure is repeated, the number of equations and the terms on the right hand side of them is reduced by one. After repetition and comparing with (11), we obtain \( m \) equations in the following form

\[
    p_m \varphi_{m-1} + p_{m-1} \varphi_{m-2} + \cdots + p_1 \varphi_{1} + \varphi_0 = 0, \quad j \in \{m, \ldots, 2m - 1\}
\]  

(14)

which provides a system of \( m \) linear simultaneous equations in terms of \( p_1, \ldots, p_m \). This system can be stated in the following matrix form

\[
    \varphi P = q
\]  

(15)

Where
\[ \Phi = \begin{bmatrix} \phi_0 & \phi_1 & \cdots & \phi_{m-1} \\ \phi_1 & \phi_2 & \cdots & \phi_m \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m-1} & \phi_m & \cdots & \phi_{2m-2} \end{bmatrix} ; \quad p = \begin{bmatrix} p_m \\ \vdots \\ p_1 \end{bmatrix} ; \quad q = \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_{2m-1} \end{bmatrix} \] 

(16)

After solving (15) for \( p \) we can find the roots of the polynomial (10) which are the values of \( t_1, \cdots, t_m \). Having these values, we can then chose any \( m \) equations from (9) to obtain \( a_1, \cdots, a_m \). For example, selecting the first \( m \) equations, we obtain the following matrix equation

\[ Ta = g \] 

(17)

where

\[ T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \\ \vdots & \vdots & \ddots & \vdots \\ t_{m-1} & t_m & \cdots & t_{2m-2} \end{bmatrix} ; \quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} ; \quad g = \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_{2m-1} \end{bmatrix} \] 

(18)
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